Cohomolgy Designs as building blocks in constructions of Weighing and Hadamard Matrices

Radel Ben-Av[1] Giora Dula[4] Assaf Goldberger[2] Yossi Strassler[3]

ArasuFest, Kalamata Aug 1 2019

[1]Holon Institue of Technology rbenav@gmail.com

[2]Tel-Aviv University assafg@post.tau.ac.il

[3] Dan Yishay danyishay@gmail.com

[4] Netanya Accademic College giora@netanya.ac.il

Introduction

Weighing matrices

Definition

A Weighing Matrix $W(N, w)$ is a $\{0, \pm 1\}$-matrix W s.t.

$$WW^T = wI_N.$$

Examples (q an odd prime power):

- Hadamard Matrices $H(N)$, $(w = N)$
- Payley Conference Matrices $W(q + 1, q)$
- Projective Spaces $W\left(\frac{q^{n+1}-1}{q-1}, q^n\right)$
- Smallest unknown: $W(35, 25)$.

Monomial Transformations and Automprphisms

Monomial Transformations

Fact: Weighing Matrices are preserved by monomial transformations:

$$X = W(n, w) \implies MXN^T = W(n, w),$$

where M, N are monomial.

Definition

A monomial pair (M, N) is an Automorphism of X is

$$MXN^T = X.$$

They form a group $\widetilde{\mathrm{Aut}}(X)$.

A Motivating Example

This is a $W(7,4)$:

$$X = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & -1 & 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & -1 & 0 & -1 & -1 \\ 0 & 1 & -1 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 \end{pmatrix}$$

A Motivating Example

This is a $W(7,4)$:

$$X = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & -1 & 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & -1 & 0 & -1 & -1 \\ 0 & 1 & -1 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 \end{pmatrix}$$

- Every (permutation) automorphism of $|X|$ (entrywise abs value) can be lifted to an automorphism of X.

A Motivating Example

This is a $W(7,4)$:

$$X = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & -1 & 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & -1 & 0 & -1 & -1 \\ 0 & 1 & -1 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 \end{pmatrix}$$

- Every (permutation) automorphism of $|X|$ (entrywise abs value) can be lifted to an automorphism of X.
- How general is this phenomena?

The ALP

The Automprphism Lifting Problem (ALP):

The ALP

Given a $\{0,1\}$-matrix $|X|$, and a subgroup $G \subseteq \widetilde{Aut}(|X|)$, find all $\{0,-1,1\}$-matrices X above $|X|$ and a subgroup $\widehat{G} \subseteq \widetilde{Aut}(X)$, mapping onto G.

Cohomology Developed Matrices

We begin with the following data:

- Let G be a finite group.
- Let X and Y be finite sets with G-action.
- Let $\mathcal{O} \subset X \times Y$ be G-stable.
- Let μ be an Abelian Group, with G action. Let $\mu^+ := \mu \cup \{0\}$.

Definition

An (μ-valued) $X \times Y$ matrix (with support $\mathcal{O}$) is a matrix

$$F = (f(x, y)) \text{ for some function } f : X \times Y \to \mu^+,$$

and $\mathrm{supp}(f) = \mathcal{O}$.

- G acts on $X \times Y$ matrices:

$$gF = \big(gf(g^{-1}x, g^{-1}y)\big).$$

Cohomology Developed Matrices

Definition

Two matrices A and B are D-equivalent if

$$A = D_1 B D_2, \; D_i \text{ diagonal.}$$

Write $A \sim_D B$.

Definition

An $X \times Y$ F is a Cohomology Developed Matrix=CDM if

$$\forall g \in G, \quad gF \sim_D F.$$

- So every $g \in G$ induces an automorphism of F.
- But Note: The notion of automorphisms includes *twisting coefficients* in μ.

Examples

- Group Developed Matrices: Take $X = Y = G$ with actions $(g, x) \mapsto xg^{-1}$ and $(g, y) = gy$, and μ with trivial action. Then $F = (f(xy))$ satisfies $gF = F$ for all $g \in G$.

Examples

- Group Developed Matrices: Take $X = Y = G$ with actions $(g, x) \mapsto xg^{-1}$ and $(g, y) = gy$, and μ with trivial action. Then $F = (f(xy))$ satisfies $gF = F$ for all $g \in G$.
- Cocyclic Matrices: Take X, Y, μ as above, and let

$$F = (f(xy)\omega(x, y)), \quad \omega : G \times G \to \mu \text{ is a 2-cocycle:}$$

$$(* - \text{cocycle condition}) \quad \omega(y, z)\omega(x, yz) = \omega(xy, z)\omega(x, y).$$

Examples

- Group Developed Matrices: Take $X = Y = G$ with actions $(g, x) \mapsto xg^{-1}$ and $(g, y) = gy$, and μ with trivial action. Then $F = (f(xy))$ satisfies $gF = F$ for all $g \in G$.
- Cocyclic Matrices: Take X, Y, μ as above, and let

$$F = (f(xy)\omega(x, y)), \quad \omega : G \times G \to \mu \text{ is a 2-cocycle:}$$

$$(* - \text{cocycle condition}) \quad \omega(y, z)\omega(x, yz) = \omega(xy, z)\omega(x, y).$$

Verification

Using the cocycle condition $(*)$:

$$\omega(xg^{-1}, gy)f(xg^{-1}gy) = \omega(g^{-1}, gy)\omega(x, y)f(xy)\omega(x, g^{-1})^{-1}$$
$$\sim_D \omega(x, y)f(xy).$$

Examples-The Fourier Matrix

- The Fourier Matrix: Let $G =$ the *Affine Group* over $\mathbb{Z}/n$:

$$G = \{x \mapsto ax + b \mid a \in (\mathbb{Z}/n)^{\times}, b \in \mathbb{Z}/n\}.$$

- Let $X = Y = \mathbb{Z}/n$ with the affine G-action.
- Let $\mu = \mu_n = \{n \text{ th roots of } 1\}$, with the following twist action:

$$(a, b)\zeta = \zeta^{a^2}.$$

- Then $F = (\exp(2\pi ixy/n))$ is a CDM:

$$(a, b)F = diag(\omega^{b^2/2-bx}) \cdot F \cdot diag(\omega^{b^2/2-by}).$$

Examples-DetFourier

- Let $X = Y = (\mathbb{Z}/n)^2$, with the action of

$$G = \{v \mapsto Av + a | A \in GL_2(\mathbb{Z}/n), a \in (\mathbb{Z}/n)^2\}.$$

- Again let $\mu = \mu_n$ with the twist action

$$(v \mapsto Av + a)\zeta = \zeta^{\det A}.$$

- Then the $n^2 \times n^2$ DetFourier Matrix given by

$$DF(u, v) = \exp(2\pi i \det(u, v)/n)$$

is a CDM. It is also Hermitian and Orthogonal.

CDMs and other Mathematics

A list of topics and extensions related to CDM's:

- Group Cohomology
- The theory of Brauer Groups
- Hecke Algebras and Representation Theory
- Association Schemes and Weights (as of D.G. Higman)
- Constructions of Weighing and Hadamard Matrices
- Simplices and MUB's in complex projective spaces
- Higher dimensional Design Theory
- (Dual theory) Magic Squares

-Will (not) be slightly discussed.

Some Theory

CDM Groups

Definition

Given G, X, Y, μ and $\mathcal{O} \subset X \times Y$ as above, let

$$CDM(G, \mathcal{O}) = \{\text{All CDM's w.r.t. } G, \mathcal{O}\}.$$

$$CDM(G, \mathcal{O})_D = CDM(G, \mathcal{O})/\text{D-equivalence}.$$

- Both sets are groups w.r.t. the Hadamard Multiplication.
- The group $CDM(G, \mathcal{O})_D$ is best described in terms of a spectral sequence (not discussed here).
- The elements of $CDM(G, \mathcal{O})$ have G as a (monomial) automorphism subgroup, whose permutation data was given in advance by X, Y and μ.

Cohomology Basics

Let G be a group, acting on a (multiplicative) Abelian group μ. For any function

$$f : G^n \to \mu,$$

let $d_i f : G^{n+1} \to \mu$ given by

$$d_i f(g_0, g_1, \ldots, g_n) = \begin{cases} f(g_0, \ldots, g_{i-1}g_i, \ldots, g_n) & i > 0 \\ g_0 f(g_1, \ldots, g_n) & i = 0. \end{cases},$$

and let

$$df = \prod_{i=0}^{n} (d_i f)^{(-1)^i}.$$

Definition

1. f is a cocycle if $df = 1$,
2. f is a coboundary if $f = dh$. Coboundaries $\subseteq$ Cocycles.

Cohomology Basics

The n-Cohomology Group is

$$H^n(G, \mu) = \frac{n - \text{cocycles}}{n - \text{coboundaries}}.$$

If H is is subgroup of G, then there is a homomorphism

$$res : H^n(G, \mu) \to H^n(H, \mu),$$

by restricting cocycles to H.

Construction of CDM's

For the sake of exposition, assume the following:

1) X, Y are G-transitive, with chosen basepoints $x_0 \in X$ and $y_0 \in Y$.
2) $\mathcal{O}$ is irreducible: If L, R are diagonal and $LAR^* = A$ for all $\mathcal{O}$-supported matrices A, then $L = R = scalar$.

The Data we shall need is:

- $G, X, Y, \mu, \mathcal{O}$ as above

Construction of CDM's

For the sake of exposition, assume the following:

1) X, Y are G-transitive, with chosen basepoints $x_0 \in X$ and $y_0 \in Y$.
2) $\mathcal{O}$ is irreducible: If L, R are diagonal and $LAR^* = A$ for all $\mathcal{O}$-supported matrices A, then $L = R = scalar$.

The Data we shall need is:

- $G, X, Y, \mu, \mathcal{O}$ as above
- The stabilizers H_X of x_0 and H_Y of y_0

Construction of CDM's

For the sake of exposition, assume the following:

1) X, Y are G-transitive, with chosen basepoints $x_0 \in X$ and $y_0 \in Y$.
2) $\mathcal{O}$ is irreducible: If L, R are diagonal and $LAR^* = A$ for all $\mathcal{O}$-supported matrices A, then $L = R = scalar$.

The Data we shall need is:

- $G, X, Y, \mu, \mathcal{O}$ as above
- The stabilizers H_X of x_0 and H_Y of y_0
- Two 1-Cocycles $\psi_X : H_X \to \mu$ and $\psi_Y : H_Y \to \mu$

Construction of CDM's

For the sake of exposition, assume the following:

1) X, Y are G-transitive, with chosen basepoints $x_0 \in X$ and $y_0 \in Y$.
2) $\mathcal{O}$ is irreducible: If L, R are diagonal and $LAR^* = A$ for all $\mathcal{O}$-supported matrices A, then $L = R = scalar$.

The Data we shall need is:

- $G, X, Y, \mu, \mathcal{O}$ as above
- The stabilizers H_X of x_0 and H_Y of y_0
- Two 1-Cocycles $\psi_X : H_X \to \mu$ and $\psi_Y : H_Y \to \mu$
- One 2-Cocycle $\omega : G \times G \to \mu$

Construction of CDM's

For the sake of exposition, assume the following:

1) X, Y are G-transitive, with chosen basepoints $x_0 \in X$ and $y_0 \in Y$.
2) $\mathcal{O}$ is irreducible: If L, R are diagonal and $LAR^* = A$ for all $\mathcal{O}$-supported matrices A, then $L = R = scalar$.

The Data we shall need is:

- $G, X, Y, \mu, \mathcal{O}$ as above
- The stabilizers H_X of x_0 and H_Y of y_0
- Two 1-Cocycles $\psi_X : H_X \to \mu$ and $\psi_Y : H_Y \to \mu$
- One 2-Cocycle $\omega : G \times G \to \mu$

It is required that *res* ω will vanish in $H^2(H_X, \mu)$ and $H^2(H_Y, \mu)$.

Construction of CDM's

- Compute trivializations λ_X and λ_Y s.t. $\omega|_{H_X \times H_X} = d\lambda_X$ and $\omega|_{H_Y \times H_Y} = d\lambda_Y$.

- For any $x \in X, y \in Y$, choose $g_x, g_y \in G$, s.t.

$$x = g_x x_0$$
$$y = g_y y_0.$$

- Define maps $for_X : G \to H_X$ and $for_Y : G \to H_Y$ by

$$for_X(g) = g \cdot g_{g^{-1}x_0},$$
$$for_Y(g) = g \cdot g_{g^{-1}y_0}.$$

Construction of CDM's

- For any $g \in G, x \in X$, let:

$$h = for_X(g_x^{-1} g).$$

- Let

$$\delta_{g,x} := g_x \left(\frac{\psi_X(h)\lambda_X(h)\omega(g_x^{-1}, g)}{\omega(h,1)} \right).$$

- Finally, let

$$D_X(g) = diag(\delta_{g,x})_x$$

and define $D_Y(g)$ similarly.

Construction of CDM's

Theorem

- The map $A \mapsto D_X(g)(gA)D_Y(g)^{-1}$ defines a monomial G-action on $\mathcal{O}$-matrices.

- An invariant matrix A is a CDM.

- All CDM's arise this way.

Theorem

There is a filtration $CDM_0 \subseteq CDM_1 \subseteq CDM_2 := CDM_D$ such that:

$$CDM_0 = \{A | gA = A\}$$

$$CDM_1/CDM_0 = \text{ A subquotient of } H^1(H_X, \mu) \oplus H^1(H_Y, \mu),$$

$$CDM_2/CDM_1 = \text{ A subgroup of } H^2(G, \mu).$$

Orientability

- Having constructed a monomial G-action, we may try to construct a matrix by Spreading:
  - Choose basepoints (orbit heads) for each G-orbit in $\mathcal{O} \subseteq X \times Y$.
  - Fix a value in μ for orbit head.
  - Use the G-action to spread the values along the orbits.

Orientability

- Having constructed a monomial G-action, we may try to construct a matrix by Spreading:
  - Choose basepoints (orbit heads) for each G-orbit in $\mathcal{O} \subseteq X \times Y$.
  - Fix a value in μ for orbit head.
  - Use the G-action to spread the values along the orbits.

Problem

The G-action may give conflicting values to some entries (orbits).
These orbits are said to be non-orientable.
Otherwise the orbit is orientable.

Example $G = B_3$

- Consider the symmetry group of the cube, B_3.
- We think of B_3 as the group of 3×3 monomial matrices.
- Let ($\mathbf{e}_i =$ the standard vectors)

$$F = \text{ Faces of the cube } = \{\pm\mathbf{e}_1, \pm\mathbf{e}_2, \pm\mathbf{e}_3\}$$

$$D = \text{ Antipodal Edges } = \{\pm\mathbf{e}_i \pm \mathbf{e}_j,\ i < j\}/\{\pm 1\}.$$

- There are two B_3-orbits in $F \times D$:

$$\begin{pmatrix} 1 & 2 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 & 2 \\ 2 & 1 & 1 & 2 & 1 & 1 \\ 2 & 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 & 2 \\ 1 & 2 & 2 & 1 & 1 & 1 \end{pmatrix}$$

Example Cont'd

- There are independent characters $\chi_1, \chi_2 : B_3 \to \mu = \{\pm 1\}$:

$$\chi_1(g) = \det(g)$$

$$\chi_2(g) = \text{ product of signs in } g.$$

- We may form a G-action by $A \mapsto \chi_1(g)\chi_2(g)(gA)$.
- Then orbit 1 is orientable and orbit 2 isn't.
- The resulting CDM is

$$\pm \begin{pmatrix} 1 & 0 & 0 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 & 1 \\ 0 & -1 & -1 & 0 & 1 & 1 \\ -1 & 1 & 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 1 & -1 & -1 \end{pmatrix}.$$

Application to Constructions of Weighing and Hadamard matrices

Projective Space Weighing Matrices

- The well known $W = W\left(\frac{q^{n+1}-1}{q-1}, q, \mu_n\right)$ weighing matrices (q a prime power, $n|q-1$) happen to be CDM's.
- In this setting we have

$$X = \text{ The points of } \mathbb{P}^n(\mathbb{F}_q),$$
$$Y = \text{ The hyperplanes of } \mathbb{P}^n(\mathbb{F}_q),$$
$$G = PGL(n+1, \mathbb{F}_q).$$

- Two orbits in $X \times Y$, by occurence relations.
- Only the non-occuring orbit is orientable.
- The associated 2-cocycle ω is the one associated with the extension

$$1 \to \mathbb{F}_q^\times \to GL(n+1, \mathbb{F}_q) \to G \to 1.$$

Projective Space Weighing Matrices

- Q: But why is W orthogonal?

Projective Space Weighing Matrices

- Q: But why is W orthogonal?
- A1: It is balanced. The products $W_{i,j} W^*_{i,k}$ are equi-distributed along μ_n.

Projective Space Weighing Matrices

- Q: But why is W orthogonal?
- A1: It is balanced. The products $W_{i,j} W^*_{i,k}$ are equi-distributed along μ_n.
- A2: WW^* is a CDM with two orbits. Only the diagonal is orientable!
- So non-orientability can be good!

Grassmannian and Flag Weighing Matrices

- By using the power of CDM theory, we may construct:

Grassmannian Weighing Matrices

Under similar conditions

$$\exists W\left(\begin{bmatrix} d \\ k \end{bmatrix}_q, q^{k(d-k)}, \mu_n\right)$$

($\begin{bmatrix} d \\ k \end{bmatrix}_q$=Gaussian Binomial Coefficients).

Flag Variety Weighing Matrices

Let $d = \sum_{i=0}^{r} k_i,\ k_i > 0$. Then

$$\exists W\left(\begin{bmatrix} d \\ k_0, k_1, \ldots, k_r \end{bmatrix}_q, q^{\sum_{i<j} k_i k_j}, \mu_n\right),$$

Provided that $n \geq r$.

Some Constructions

The Single/Double Circulant Core Structure:

$$\left[\begin{array}{c|c|c|c}
 & & \mathrm{a} & \mathrm{b} \\
A & B & \vdots & \vdots \\
 & & \mathrm{a} & \mathrm{b} \\
\hline
 & & \mathrm{c} & \mathrm{d} \\
C & D & \vdots & \vdots \\
 & & c & d \\
\hline
e \cdots e & f \cdots f & p & q \\
\hline
g \cdots g & h \cdots h & r & s
\end{array}\right]$$

$$G = \mathbb{Z}/n\mathbb{Z}, \quad X = Y = G \cup G \cup \{*\} \cup \{*\}$$

Single Negacyclic Core of order $2n$ (n-odd)

$$\left[\begin{array}{cccccc|cc}
a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a & b \\
-a_6 & a_1 & a_2 & a_3 & a_4 & a_5 & b & -a \\
-a_5 & -a_6 & a_1 & a_2 & a_3 & a_4 & -a & -b \\
-a_4 & -a_5 & -a_6 & a_1 & a_2 & a_3 & -b & a \\
-a_3 & -a_4 & -a_5 & -a_6 & a_1 & a_2 & a & b \\
-a_2 & -a_3 & -a_4 & -a_5 & -a_6 & a_1 & b & -a \\
\hline
c & d & -c & -d & c & -d & e & f \\
-d & c & d & -c & -d & c & -f & e
\end{array}\right]$$

$$G = \mathbb{Z}/n\mathbb{Z}, \quad X = Y = \mathbb{Z}/n\mathbb{Z} \ \cup \ \mathbb{Z}/2,$$

Negacyclic action. Such matrices form an Algebra.

This is an $OD(12,6,6)$

$$\left[\begin{array}{rrrrrr|rrrrrr}
a & -b & -b & b & b & b & -b & -a & -a & a & a & a \\
-b & a & b & -b & b & b & -a & -b & a & -a & a & a \\
-b & b & a & b & -b & b & -a & a & -b & a & -a & a \\
b & -b & b & a & -b & b & a & -a & a & -b & -a & a \\
b & b & -b & -b & a & b & a & a & -a & -a & -b & a \\
b & b & b & b & b & a & a & a & a & a & a & -b \\
\hline
b & a & a & -a & -a & -a & a & -b & -b & b & b & b \\
a & b & -a & a & -a & -a & -b & a & b & -b & b & b \\
a & -a & b & -a & a & -a & -b & b & a & b & -b & b \\
-a & a & -a & b & a & -a & b & -b & b & a & -b & b \\
-a & -a & a & a & b & -a & b & b & -b & -b & a & b \\
-a & -a & -a & -a & -a & b & b & b & b & b & b & a
\end{array}\right]$$

$$G \simeq A_5 \times \mathbb{Z}/2\mathbb{Z}, \quad X = Y = G/D_5 \times \{0\}$$

Such matrices generate over $\mathbb{Q}$ the field $\mathbb{Q}(\sqrt{5}, \sqrt{-1})$.

Orthogonal Pairs

Definition

1) Two (possibly rectangular) CDM's A, B are an Orthogonal Pair (OP) if AB^* is nowhere orienatble.
2) A tuple of CDM's $(A_1, \ldots, A_m)$ is an Orthogonal Tuple if each (A_i, A_j) is an OP.

- If (A, B) is an OP, then necessarily $AB^* = 0$.
- Orthogonality is a consequence of the group action. So one obtains *symbolic* (A, B) which are formally orthogonal.
- One may substitute (rectangular) blocks in place of symbols. No commutativity/amicability restrictions.

Here is an example of an OP:

$$A = \begin{pmatrix} -a & -b & a & -b & -a & -b & b & b & -b & a & b & b \\ -b & a & b & b & b & -b & -b & b & -a & -b & a & -a \\ -b & b & -b & -a & b & a & -a & a & -b & b & -b & b \\ -b & b & -b & a & b & -a & a & -a & -b & b & -b & b \\ -a & b & a & b & -a & b & -b & -b & b & a & -b & -b \\ -b & -a & b & b & b & -b & -b & b & a & -b & -a & a \end{pmatrix}$$

$$B = \begin{pmatrix} c & -d & -c & -d & -c & -d & d & d & -d & c & d & d \\ -d & c & d & -d & d & -d & d & d & c & -d & -c & -c \\ d & d & -d & -c & -d & -c & c & c & d & d & -d & -d \\ -d & -d & d & c & d & c & -c & -c & -d & -d & d & d \\ -c & d & c & d & c & d & -d & -d & d & -c & -d & -d \\ d & -c & -d & d & -d & d & -d & -d & -c & d & c & c \end{pmatrix}.$$

This is a $W(16,8)$ constructed from OT's. Each 16×8 column is an OT.

$$
\left(\begin{array}{rrrrrrrr|rrrrrrrr}
1 & 0 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 1 & 1 \\
-1 & 0 & -1 & 0 & 0 & 0 & 1 & -1 & 0 & -1 & 0 & -1 & 1 & 1 & 0 & 0 \\
0 & -1 & 0 & 1 & 1 & 1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 1 & 1 \\
0 & -1 & 0 & 1 & 0 & 0 & 1 & -1 & 1 & 0 & 1 & 0 & -1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\
-1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 \\
\hline
0 & -1 & 0 & -1 & 0 & 0 & -1 & -1 & 1 & 0 & -1 & 0 & -1 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & -1 & 1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\
1 & 0 & -1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & -1 & -1 & 1 & 0 & 0 \\
-1 & 0 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 1 & -1 \\
-1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 1 \\
0 & 0 & 0 & 0 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\
\hline
1 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & -1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\
\hline
-1 & -1 & 0 & 0 & -1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & -1 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0
\end{array}\right)
$$

$G = B_3$ (Symmetry Group of the Cube)

Sets: $X_1 = X_2 =$ Edges$/\{\pm 1\}$, $X_3 = X_4 = $ *Orientations*

$Y_1 = Y_2 =$Vertices,

Cohomology Classes: trivial 2-cocycle, nontrivial 1-cocycles.

A similar Hadamard 16 example (Same Group, different Sets)

$$
\left(\begin{array}{rrrrrrrr|rrrrrrrr}
1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 \\
-1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 \\
1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 \\
1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 \\
-1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 \\
-1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 & 1 \\
\hline
1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 \\
-1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & -1 & 1 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & -1 & -1 \\
1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 \\
1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 \\
\hline
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 \\
-1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 \\
\hline
1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & 1
\end{array}\right)
$$