Reconstructing Weighing Matrices from Their Automorphism Group

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Definitions and Properties

Definition

A Weighing Matrix is an $n \times n$ $\{0, -1, 1\}$-matrix W such that

$$WW^T = kI_n.$$

We then say that W is a $W(n, k)$.

Definition

Two $\{0, -1, 1\}$-matrices W, W' are isomorphic if one can write

$$PWQ^T = W'$$

for square monomial $\{0, -1, 1\}$-matrices P and Q.
We call the pair (P, Q) the isomorphism. We write

$$W \simeq W'.$$

Definitions and Properties

- Compsition of isomorphisms is given by $(P_1, Q_1) \circ (P_2, Q_2) = (P_1 P_2, Q_1 Q_2)$.
- If W is a $W(n, k)$ and $W \simeq W'$ then W' is a $W(n, k)$.

Definition

For a $\{0, -1, 1\}$-matrix W, the set

$$Aut(W)^{\sim} = \left\{ (P, Q) |\ PWQ^T = W;\ P, Q \text{ monomial} \right\} \quad (1)$$

$$Aut(W) = Aut(W)^{\sim} / \langle (I, I) \rangle \quad (2)$$

is a group under composition, called the Automorphism Group of W. We also define a subgroup $PermAut(W) \subseteq Aut(W)$ by

$$PermAut(W) = \left\{ (P, Q) \in Aut(W)^{\sim} |\ P, Q \text{ are permutations} \right\}.$$

Main problem and Motivation

Main Problem

- Can we reconstruct W from $Aut(W)$?
- More generally: Given a finite group $G \subseteq S_n \times S_n$, find a $\{0, -1, 1\}$-$n \times n$ matrix W and an injection $G \hookrightarrow Aut(W)$.

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- Often we obtain families thereof.

Disadvantage

- Poorly symmetric weighing matrices will not be caught by such methods.

Example

This is a $W(13,9)$:

$$W = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & -1 & -1 & -1 & 1 & 1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 \\
1 & -1 & -1 & 1 & 0 & 0 & -1 & 1 & 0 & 1 & 1 & 0 & -1 \\
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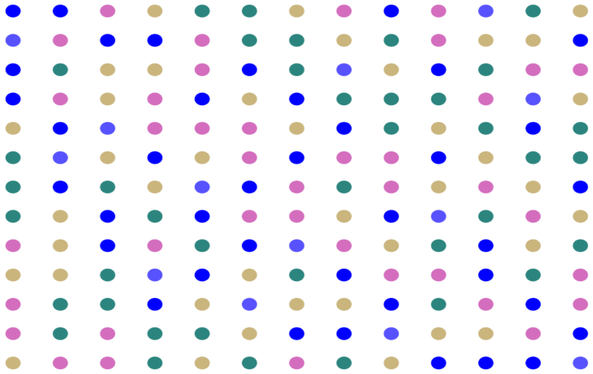
0 & 1 & 0 & -1 & -1 & 1 & -1 & 1 & 0 & 0 & 1 & -1 & 1 \\
0 & 0 & 1 & -1 & 1 & -1 & 0 & 1 & -1 & -1 & 1 & 0 & -1
\end{pmatrix}$$

Example

Here is how $Aut(W)$ breaks W into orbits:



- There are 5 orbits of orders $39, 39, 39, 39, 13$.
- Each orbit is determined via $Aut(W)$ from a single entry.
- $\implies$ Can recover W from at most 3^5 candidates.

- For a $\{0,-1,1\}$-matrix W, let $|W|$ denote the elementwise absolute-value of W.
- There is a natural homomorphism

$$Aut(W) \xrightarrow{abs} PermAut(|W|).$$

- In the example above, $PermAut(|W|) \simeq PGL_3(\mathbb{F}_3)$, while $Aut(W) \simeq (\mathbb{Z}_{13} \ltimes \mathbb{Z}_3)$. *abs* is an injection.

A Projective-Plane Example

Consider another $W(13, 9)$:

$$W = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & -1 & 1 & 1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 \\ 1 & -1 & 1 & 0 & -1 & 0 & -1 & 1 & 0 & -1 & 1 & 0 & -1 \\ 1 & -1 & 0 & 1 & 0 & -1 & 1 & -1 & 0 & -1 & 0 & 1 & 1 \\ 1 & 1 & -1 & 0 & 0 & -1 & -1 & 0 & 1 & 0 & 1 & -1 & 1 \\ 1 & 1 & 0 & -1 & -1 & 0 & 0 & -1 & 1 & 0 & -1 & 1 & -1 \\ 1 & 0 & -1 & 1 & -1 & 0 & 1 & 0 & -1 & 1 & 0 & -1 & -1 \\ 1 & 0 & 1 & -1 & 0 & -1 & 0 & 1 & -1 & 1 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & 0 \\ 0 & 1 & -1 & 0 & -1 & 1 & 0 & 1 & -1 & -1 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 & 1 & -1 & 1 & 0 & -1 & -1 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 & -1 & 1 & 1 & -1 & 0 & 0 & 1 & -1 & 1 \end{pmatrix}$$

- $|W|$ is the anti-incidence matrix of the projective plane $\mathbb{P}^2(\mathbb{F}_3)$.
- This time $\#Aut(W) = 5616$. Moreover,

$$Aut(W) \xrightarrow{\simeq} PermAut(|W|) \text{ is an isomorphism!}$$

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- Is this the case for all projective spaces?
- Answer: Yes! (and more)

The kernel and cokernel of abs

Definition

A matrix W is called irreducible if $W \not\simeq \bigoplus_i W_i$ for smaller matrices W_i.

Theorem

If W is nonsingular and irreducible, then

$$abs: \; Aut(W) \hookrightarrow PermAut(W)$$

is an injection.

- In the following, we will discuss the image of *abs*.

Definitions and Notation

- Let S_n be the group of $n \times n$ permutation matrices.
- Let B_n be the group of $n \times n$ sign monomial matrices. We have the split exact sequence

$$1 \longrightarrow \{\pm 1\}^n \longrightarrow B_n \xrightarrow{\pi} S_n \longrightarrow 1 \quad .$$

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$$\begin{array}{ccccccccc} 1 & \longrightarrow & \{\pm 1\}^n & \longrightarrow & B_n & \xrightarrow{\pi} & S_n & \longrightarrow & 1 \\ & & & & & {\color{red}{\nwarrow s}} & \uparrow & & \\ & & & & & & G & & \end{array} .$$

Definition

Let $G \subseteq S_n$ be a subgroup. A coloring for G is a section $s : G \to B_n$ such that $\pi \circ s = id_G$.

- In other words, $G' = s(G) \subset B_n$ maps isomorphically onto G.

Definitions and Notation

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An abstract Automorphism group is a finite group G, together with two embeddings $L', R' : G \hookrightarrow B_n$.

- We say that (L', R') is a coloring of (L, R) if

$$L = abs \circ L' \quad \text{and} \quad R = abs \circ R'.$$

Non Conflicting Orbits

- Let $I = \{1, 2, \ldots, n\}$.
- An abstract PermAutomorphism group $L, R : G \hookrightarrow S_n$ acts on pairs $(i, j) \in I \times I$ by
$$g(i, j) \;=\; (L(g)i, R(g)j)\,.$$

Definition

Let $L', R' : G \hookrightarrow B_n$ be a coloring of (L, R). An orbit in $I \times I$ is Non-Conflicting, if

$$\text{sign}\ \ L'(g)_{i,i} = \text{sign}\ \ R'(g)_{j,j}\ \forall g, i, j \text{ such that } g(i, j) = (i, j).$$

- It is enough to check this property for just one (i, j) in an orbit.

- For each noncolicting orbit O, fix $(i_O, j_O) \in O$.

Theorem

For any Abstract Automorphism Group $L', R' : G \hookrightarrow B_n$, and any choice of $\mu_O \in \{0, -1, 1\}$, O nonconflicting, there exists a unique $\{0, -1, 1\}$-matrix $W \neq 0$ such that

(i) $A_{i_O, j_O} = \mu_O$ *for all nonconflicting* O,

(ii) $A_{i,j} = 0$ *if* (i, j) *belongs to a conflicting orbit,*

(iii) $G \subseteq Aut(W)$ *via* (L', R').

Observation

Suppose further that $L(G) \subset S_n$ is 2-transitive. Then $|WW^T|$ is constant off the diagonal. In particular, there are **good** chances that W will be a weighing matrix!

- The more nonconflicting G-orbits of $I \times I$, the more candidates for W to be a weighing matrix.
- The more transitive is $L(G)$ (or $R(G)$), the better chance for W to be a weighing matrix.
- Sometimes we get a matrix W which can be augmented to $\bar{W}$, a weighing matrix.
- In the 2-transitive case, we get balanced weighing matrices.

Coloring and Cohomology

- Let $G \subset S_n$ be a subgroup. Write $N = \{\pm 1\}^n$. Then G acts on N by permutations (written $n \mapsto n^g$).

Definition

A crossed homomorphism or a 1-cocycle $x : G \to N$ is a map such that

$$x(gg') = x(g) \cdot x(g')^g.$$

A coboundary is a map $c : G \to N$ such that

$$c(g) = n^g \cdot n^{-1}.$$

Coloring and Cohomology

- Two crossed homomorphisms x_1, x_2 are cohomologous if

$$x_1(g)x_2(g)^{-1} = c(g)$$

for a coboundary $c(g)$. This is an equivalence relation. Write $x_1 \approx x_2$.

Definition

We define the 1st cohomology group as

$$H^1(G,N) = \{\text{Crossed Homomorphisms} : G \to N\}/\approx .$$

- There is a bijection

$$H^1(G,N) \leftrightarrow \{\text{Colorings } G \to B_n \text{ up to conjugation by } N\}$$

Shapiro's Lemma

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Suppose that $G \hookrightarrow S_n$ acts transitively on I. Then there is an isomorphism

$$H^1(G, N) \simeq Hom(H, \{\pm 1\}),$$

where $H = Stab(pt)$ is the stabilizer of some point $pt \in I$.

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where $H = Stab(pt)$ is the stabilizer of some point $pt \in I$.

1. Fix a set of representatives for $H \backslash G$. Write $Hc = H\overline{c}$ for a representative $\overline{c}$.
2. For a class $[x] \in H^1(G,N)$, let $\psi : H \to \{\pm 1\}$ be $\psi(h) = x(h)_{pt}$.
3. For $\psi : H \to \{\pm 1\}$, match $x : G \to N$ such that

$$x(g)_{i=g_2^{-1}(pt)} := \psi(\overline{g_2} g \overline{g_2 g}^{-1}).$$

This process is called Induction.

The Algorithm

Given embeddings $L, R : G \hookrightarrow S_n$ as transitive subgroups:

1. Find two homorphisms $\psi_1, \psi_2 : H_1, H_2 \to \{\pm 1\}$.
2. Obtain by Induction two classes $[x_1], [x_2] \in H^1(G, N)$.
3. Generate the liftings $L', R' : G \to B_n$.
4. Find the nonconflicting orbits.
5. For each nonconflicting orbit, put $0, \pm 1$ at a single position and complete by the action of G. Put 0 othwerwise.
6. If a weighing matrix, stop. Else, repeat step 5.

Remark

Replacing x_1, x_2 by cohomologous x_1', x_2' results with an isomorphic matrix.

Condition for nonconflictness

Theorem

- *Let* $L, R : G \to S_n$ *be a transitive abstract PermAutomorphism.*
- *Let* $(i, j) \in I \times I$ *with* $H_1 = Stab_G(i)$ *and* $H_2 = Stab_G(j)$*.*
- *Let* $\psi_i : H_i \to \{\pm 1\}$ *be homomorphisms.*
- *Let* $(i_1, j_1) \in I \times I$ *be arbitrary, with* $i_1 = L(g_1)i$ *and* $j_1 = L(g_2)j$*.*

The (i_1, j_1) *is nonconflicting, if and only if*

$$\psi_1(g_1^{-1} g g_1) = \psi_2(g_2^{-1} g g_2) \;\; \forall \; g \in g_1 H_1 g_1^{-1} \cap g_2 H_2 g_2^{-1}.$$

Example: Affine groups

- Let $\mathbb{F}_q$ be a finite field of order q, and let G be the group of affine transformations on $\mathbb{F}_q$:

$$G = \left\{x \mapsto ax + b \mid a \in \mathbb{F}_q^\times, b \in \mathbb{F}_q\right\}.$$

- Take $L = R : G \hookrightarrow S_q$, in the obvious way. Action is bitransitive.
- Two Orbits: Diagonal and Off-Diagonal.
- Let $H = Stab(0) = \{x \mapsto ax\}$. Choose $\psi_1, \psi_2 : H \to \{\pm 1\}$

$$\psi_1(x \mapsto ax) = \left(\frac{a}{q}\right); \quad \psi_2(x \mapsto ax) = 1.$$

- Resulting matrix W satisfies $WW^T = qI - J$ $(J = (1)_{i,j})$
- By adding margins, we obtain Payley's $W(q+1, q)$.

Projective Spaces

- Let $V = \mathbb{F}_q^{n+1}$ be a vector space of dimension $n+1$ over $\mathbb{F}_q$.
- The Projective Space is the set

$$\mathbb{P}(V) \; = \{\text{Lines in } V \text{ through the origin}\}$$

- Fix a nondegenerate $\mathbb{F}_q$ bilinear form $\langle , \rangle$ on V.
- The Anti -Incidence matrix of $\mathbb{P}(V)$ is the matrix $|W|$ indexed by $\mathbb{P}(V)$ such that

$$|W|_{span(v),span(w)} = \begin{cases} 1 & \text{if } \langle v, w\rangle \neq 0 \\ 0 & \text{otherwise} \end{cases}.$$

- The group $PGL(V)$ acts on $\mathbb{P}(V)$. Denote this by $g \mapsto \pi(g) \in S_{\mathbb{P}(V)}$.
- Then $PGL(V) \hookrightarrow PermAut(|W|)$ by

$$\pi(g)|W|\pi({g^{-1}}^T)^T = |W|.$$

Coloring Projective Spaces

- Let $H \subset PGL(V)$ be the stabilizer of $span((1,0,\ldots,0))$.
- Then
$$H = \left\{ h = \begin{pmatrix} a & b \\ 0 & C \end{pmatrix} \middle| C \text{ is } n \times n \right\}.$$
- Take $\psi_1 = \psi_2 : H \to \{\pm 1\}$ by
$$h \mapsto \begin{cases} \left(\frac{\det h}{q}\right) & n \text{ odd } q \text{ odd} \\ \left(\frac{\det C}{q}\right) & n \text{ even } q \text{ odd} \end{cases}$$
- The resulting matrix W is a $W\left(\frac{q^{n+1}-1}{q-1}, q^n\right)$ q odd.
- $Aut(W) = PGL(n+1,q) \rtimes Gal(\mathbb{F}_q)$.

Comments

- The construction is not valid for even $q > 2$. W (if exists) must have 'less' automorphisms than has $|W|$.
- The orthogonality of W boils down to

$$\sum_{a \in \mathbb{F}_q^{\times}} \left(\frac{a}{q} \right) = 0.$$

- W also a circulant structure for n even, and negacyclic structure for n odd.
- This is naturally generallized to matrices with μ_n-entries.

Grassmannian Varieties

Definition

Fix a finite field $\mathbb{F}_q$ and integers $n > k > 0$. The (n, k) Grassmannian over $\mathbb{F}_q$ is

$$Gr(n, k; \mathbb{F}_q) \ := \ \{V \mid V \subset \mathbb{F}_q^n \text{ is a linear subspace, } \dim V = k\}.$$

One has

$$\#Gr(n, k; \mathbb{F}_q) = \begin{bmatrix} n \\ k \end{bmatrix}_q \text{ (Gaussian Binomial Coefficients)}$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q \ := \ \frac{n!_q}{k!_q(n-k)!_q}, \text{ for } m!_q \ := \ \prod_{i=1}^{m} \frac{(q^i - 1)}{q - 1}.$$

Grassmannian Varieties

Theorem

For every odd prime-power q and integers $n > k > 0$, there exists a weighing matrix

$$G_{n,k} = W\left(\begin{bmatrix} n \\ k \end{bmatrix}_q, q^{k(n-k)}\right)$$

Based on the Grasmannian $Gr(n, k; \mathbb{F}_q)$. One has

$$Aut(G_{n,k}) = PermAut(|G_{n,k}|) = PGL(n, q) \rtimes Gal(\mathbb{F}_q).$$

Comments

- The case $k = 1$ is the case of Projective Spaces.
- $G_{n,k}$ is generally not a circulant nor a group developed matrix.
- There exists a similar construction for Flag Varieties. It carries a similar automorphism group. However, they usually fail to be weighing matrices.

Quasi-Projective Geometries

- For a finite-field F, and $r|(\#F-1)$. let

$$\mathbb{P}^d(F)'_r \ := \ \left(F^{d+1}-\{0\}\right)/(F^\times)^r.$$

- Let $W = W(n,w)$ be a r-block-circulant weighing matrix.
- We get a family of

$$W\left(\frac{q^{d+1}-1}{q-1}n, q^d w\right).$$

- Not equivalent to a tensor $W\left(\frac{q^{d+1}-1}{q-1}, q^d\right) \otimes W(n,w)$, nor weaving.

QUESTIONS ?