Usage of (Multiple) Code Invariant to Find the Symmetric W(23, 16)

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Weighing matrices

- A weighing matrix $W(n, w)$ is a $n \times n$ matrix W whose elements are $0, \pm 1$ such that $WW^T = wI_n$. $W(n, w)$ denotes both a single matrix and the class of all $W(n, w)$.
- The following are $W(2, w), 1 \le w \le 2$

$$\left(\begin{array}{c|cc} I_2 & \begin{matrix} 1 & 1 \\ 1 & - \end{matrix} \end{array}\right)$$

- The following are $W(3, w), 1 \le w \le 3$

$$\left(\begin{array}{c|c|c} I_3 & None & None \end{array}\right)$$

- The following are $W(4, w), 1 \le w \le 4$

$$\left(\begin{array}{cccc|cccc|cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & - & 0 & 0 & 1 & 0 & 1 & - & 1 & - & 1 & - \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & - & 0 & 1 & 1 & 1 & - & - \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & - & - & - & 1 & 0 & 1 & - & - & 1 \end{array}\right)$$

Questions

- Applications: Chemistry, Spectroscopy, Quantum Computing and Coding Theory.
- For which n and w $W(n, w) \neq \emptyset$ is an open question.
- Hadamard conjecture: $W(n, n) \neq \emptyset$ for every $n = 4k, k \in \mathbb{N}$.
- The main mathematical interest is to exhibit a concrete $W(n, w)$ or to prove that it does not exist.
- To date the smallest Hadamard matrix whose existence is unknown is $H(668)$.
- Given $W(n, w)$ it is a mathematical interest to find if an (anti)symmetric $W(n, w)$ exists.
- In this note we present a concrete symmetric $W(23, 16)$ derived from $W(23, 16)$ found recently.

Isomorphic (Hadamard equivalent) weighing matrices

- A monomial matrix (a signed permutation) P is a permutation matrix whose non zero elements are ± 1.
- Two matrices U, V are isomorphic (Hadamard equivalent) if there exists two monomial matrices P, Q with $PUQ^t = V$.
- The following exhibits an Hadamard equivalence between the Sylvester matrix $H2 \otimes H2$ and the circulant matrix $CH4$.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & - \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & - & 1 & - \\ 1 & 1 & - & - \\ 1 & - & - & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ - & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} - & 1 & 1 & 1 \\ 1 & - & 1 & 1 \\ 1 & 1 & - & 1 \\ 1 & 1 & 1 & - \end{bmatrix}$$

Classifying equivalence classes

- All Hadamard matrices of order n with $n \leq 12$ have one Hadamard equivalence class. There are 5 Hadamard equivalence classess of Hadamard matrices of order 16, 3 for n=20 and 60 for n=24 .
- Chan Rodger Seberry classification. [CRS] classified (up to Hadamard equivalence) all weighing matrices of weight $w \leq 5$ and all weighing matrices of order $n \leq 11$. They used what is called in Assaf's paper [G] the support geometry.
- Harada Munemasa classification [HM] they classify all weighing matrices of order $n \leq 15$, $n = 17$ and all $W(16, w), w = 6, 9, 12$ and $W(18, 9)$. For example they found 11891 classes of $W(18, 9)$
- In this work we construct an invariant of the equivalence class and use it to find isomorphisms between weighing matrices.
- The isomorphisms between $W(23, 16)$ and $W(23, 16)^T$ are used to find a symmetric representative within this class.

An example

- We will illustrate our definitions on the following matrices:
- Let W be the $W(7,4)$ weighing matrix (Taken from Wikipedia, and altered)

$$W = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & -1 & 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 \end{pmatrix}$$

- Consider the submatrix $Ro = W[2,1,0]$:

$$\begin{pmatrix} 1 & 0 & -1 & 0 & -1 & 0 & 1 \\ 1 & -1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

The orbit of Ro

- We define an action by $Ro \mapsto LRoR^T$, where L is a 3×3 and R 7×7 monomial matrices.
- There are $2^3 \cdot 3! = 48$ monomial matrices of length 3, denoted L and $2^7 \cdot 7! = 645120$ monomial 7×7 matrices denoted R and they present $48 \cdot 645120 = 30965760$ pairs. The two pairs $\pm(L, R)$ induce the same isomorphism on Ro.
- We computed by exhaustive enumeration, the size of the orbit of Ro relative to this action, and it turns out, that twelve pairs (6 isomorphisms) (L, R) leave Ro unchanged, and we think of them as automorphisms of Ro.

The automorphisms of *Ro*

- We conclude that the orbit of *Ro* has size $30965760/12 = 2580480$.
- Here is one such automorphism.

$$L, R = \left[\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \right.$$

The code of a vector and of a matrix

- First, let us define $v = [1, 3, 9, \ldots, 3^{m-1}]$.
- The code of a vector $w \in \{\pm 1, 0\}^m$ is $Code(w) = v \cdot w^T$. Code is a bijection $\{\pm 1, 0\}^m \to [-\frac{3^m-1}{2}, \frac{3^m-1}{2}]$.
- The code of $w = (1, 1, 1)$ equals 13.
- Given $A \in M_{m \times n}(\{\pm 1, 0\})$ define $Code(A) = vA$,
- The code is a bijection

$$Code : M_{m \times n}(\{\pm 1, 0\}) \to \left[-\frac{3^m - 1}{2}, \frac{3^m - 1}{2}\right]^n$$

.

- $$Code(Ro) = [13, 6, 8, 9, 2, 3, 1].$$
- $Code(A)$ is not an invariant of the equivalnce class of A.
- One way to define an invariant is to compute $Code(A')$ for all elements A' in the class, and then take the 'minimum' with respect to some ordering.

projecting into the subset Norm

- an Hadamard operation is an action $Mon_{m\times m}(\{\pm 1\}) \times M_{m\times n}(\{\pm 1, 0\}) \times Mon_{n\times n}(\{\pm 1\}) \to M_{m\times n}(\{\pm 1, 0\})$.
- The size of an orbit might be $2^m m! \times 2^n n!$.
- We define a subset $Norm \subset M_{m\times n}(\{\pm 1, 0\})$ and a projection defined in two stages.
  - $Normalization : M_{m\times n}(\{\pm 1, 0\}) \twoheadrightarrow Norm$.
  - $Sort : Norm \twoheadrightarrow Norm$
- $Can = Sort \circ Normalization$ is a projection. $M_{m\times n}(\{\pm 1, 0\}) \twoheadrightarrow Norm$
- It holds that $code(Can(A)) = code(Can(AR^T))$ for all monomial matrices R
- Thus canonization reduces the enumeration over all pairs (L, R) to an enumeration over all martices L. This is a reduction factor of $2^n n!/(mn^2 \log n)$.

The Normalization Projection

- Every nonzero column has a top nonzero element which is ± 1.
- Multiplying with this leading term gives a column whose leading term is $+1$.
- Normalization changes the matrix and its code.
- The Normaliation needs an enumeration of magnitude mn.
- After the normalization the enumeration on R are of order of magnitude $n!$ instead of $2^n n!$.
- The factor of enumeration saved is $\frac{2^n}{mn}$. For *Ro* it is more than 6. For $m = 3, n = 23$ it is more than 2^{16}.
- Comment: If we replace v with $mi = -[3^{m-1}, \ldots, 1]$, the effect of *Normalize* on codes is by $z \mapsto -|z|$.

An example : the Normalization of *Ro*

$$\begin{tikzcd}
\begin{pmatrix} 1 & 0 & -1 & 0 & -1 & 0 & 1 \\ 1 & -1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix} \arrow[r, "\text{Normalize}"] \arrow[d, "\text{Code}"] & \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & -1 & 1 & 0 \\ 1 & -1 & -1 & 1 & 0 & 0 & 0 \end{pmatrix} \arrow[d, "\text{Code}"] \\
{[13, 6, 8, 9, 2, 3, 1]} \arrow[r, "\text{Normalize}"] & {[13, -6, -8, 9, -2, 3, 1]}
\end{tikzcd}$$

The Sorting Projection

- The code of the normalized matrix is an element of $[-\frac{3^m-1}{2}, \frac{3^m-1}{2}]^n$.
- Sorting changes the matrix and its code. Normalized code and matrix stay normalized after sorting.
- The sorting is usually a library package of any scientific programming language of magnitude $nln(n)$.
- $Sort \circ Normalize$ is a projection called canonization and denoted Can. $Can(A)$ is isomorphic to A.
- Assuming normalization with mi above, it holds that $\min_{R \in Mon_{n\times n}(\{\pm 1\})} Code(AR^T) = Code(Can(A))$.
- Sort saves a factor of enumeration $\frac{n!}{nln(n)}$. For Ro it is more than 360. For $m = 3, n = 23$ it is more than $\frac{22!}{5}$.

An example : the Sorting of the Normalized *Ro*

$$
\begin{tikzcd}
\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & -1 & 1 & 0 \\ 1 & -1 & -1 & 1 & 0 & 0 & 0 \end{pmatrix} \arrow[r, "\text{Sort}"] \arrow[d, "\text{Code}"] & \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & 0 & 1 \\ -1 & -1 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \arrow[d, "\text{Code}"] \\
{[13, -6, -8, 9, -2, 3, 1]} \arrow[r, "\text{Sort}"] & {[-8, -6, -2, 1, 3, 9, 13]}
\end{tikzcd}
$$

.

The code invariant of a rectangular matrix

- Given a matrix A we define the code invariant of A by

$$CI(A) = \min_{L \in Mon_{m\times m}(\{\pm 1\})} Code((Can(LA))).$$

- For A and B of the same dimensions, $CI(A) = CI(B)$ if and only if A and B are Hadamard equivalent.
- $CI(Ro) = [-11, -6, 1, 3, 4, 9, 10]$.
- The calculation of $CI(A)$ has complexity $2^m m! n^2 m ln(n)$ instead of $2^{n+m} n! m!$ and presents a vast reduction.
- Still, if m is large $CI(A)$ can not be calculated practically.

The multiple code invariant of a matrix

- Given $A \in M_{m \times n}(\{\pm 1, 0\})$ it might be impracticle to calculate $CI(A)$.
- So choose $d \leq m$. There are $\binom{m}{d}$ subsets of $\{1, 2, \cdots m\}$ of cardinality d. Each such subset determines a submatrix $S(A) \subset A$ with d rows.
- Each of the above $S(A)$ has a computable CI.
- Define the multiple code invariant of A and d, $MCI(A, d)$, as the multiset of the CI's of the above submatrices.
- $MCI(A, d)$ is an element of $\left([-\frac{3^d-1}{2}, \frac{3^d-1}{2}]^n\right)^{\binom{m}{d}}$.
- If A and B are Hadamard equivalent, every subset of d rows of A maps to a subset of d rows of B, and thus have the same CI so that $MCI(A, d) = MCI(B, d)$.

some numerical values for some $MCI(A, d)$

- For $A = W(7,4)$ mentioned above, it holds that $MCI(W,3) = \{[-11,-6,1,3,4,9,10] \times 28, [-8,-6,-2,0,4,10,12] \times 7\}$.
- In the multiset with $\binom{7}{3} = 35$ CI's 28 terms have $CI = [-11,-6,1,3,4,9,10]$ and 7 terms have $CI = [-8,-6,-2,0,4,10,12]$
- for the same W, $MCI(W,4) = \{[-35,-24,-18,1,7,12,31] \times 28, [-38,-24,-18,-6,4,10,28] \times 7\}$.

More on MCI($W(7,4)$)

Let $W = W(7,4)$

- The fact that $MCI(W,3)$ contained only two distinct CI's deserves explanation.
- The $\{0,1\}$-matrix $DG = J - |W|$ ([G], [SD],[SM]) is the incidence matrix of the projective plane (Fano)

$$X = \mathbb{P}^2(\mathbb{F}_2).$$

- There are exactly two different configuration for 3 lines in a plane:
  - Star Like: All 3 lines meet at a point;
  - Triangle like: No common point for the 3 lines.
- Hence $MCI(DG,3)$ will contain two distinct CI's

More on MCI($W(7,4)$)

- The same observations is true for $PG = |W|$ since 4 lines in X determine the complementary 3 lines.
- Fact: Every automorphism of PG can be extended uniquely (up to a total sign) to an automprhism of W.
- This is a rare phenomena, but there are some well known families of weighing matrices having this property.
- This explains why $MCI(W, 3)$ has the same multiplicities as $MCI(PG, 3)$ and $MCI(DG, 3)$.
- By the same reasoning, $MCI(W, d)$ has the same multiplicities as $MCI(W, 7 - d)$.

$MCI(W(23,16))$

Let

$$\begin{aligned} W_1 &= W(23,16) \text{ found by the 'Shaddow Geometry' method;[G]} \\ W_2 &= W(23,16) \text{ found by the 'Tiling' method([SD], [SM])} \end{aligned}$$

- By contrast, the multiplicities of $MCI(W_1, 3) = MCI(W_1^T, 3)$ are quite scattered ($m \times f$ means m items of frequency f).

$$2{\times}1, 2{\times}3, 4, 6{\times}6, 2{\times}12, 13, 2{\times}15, 18, 21, 24, 33, 48, 3{\times}54, 2{\times}57,$$
$$60, 66, 2 \times 72, 87, 102\ 120, 141, 186, 330$$

- It can be shown that W_1 has only 3 automorphisms.
- For W_2, the list of frequencies is

$$1{\times}1, 5{\times}5, 1{\times}10, 5{\times}15, 4{\times}20, 2{\times}30, 1{\times}35, 1{\times}45, 3{\times}50, 2{\times}55,$$
$$1{\times}75, 1{\times}85, 1{\times}105, 1{\times}115, 1{\times}120, 1{\times}125, 1{\times}135, 1{\times}420$$

More on $MCI(W(23, 16))$

- Similarly for W_2^T, the list of frequencies is

$$1{\times}1, 3{\times}5, 4{\times}10, 5{\times}15, 1{\times}20, 2{\times}30, 1{\times}35, 3{\times}40, 1{\times}45, 2{\times}50,$$
$$1 \times 55, 2 \times 75, 1 \times 85, 1 \times 90, 2 \times 105, 1 \times 125, 1 \times 130, 1 \times 415$$

- The fact that most frequencies in $MCI(W_2)$ are divisible by 5 reflects the fact that the automorphism group is of order 5. A similar phenomena holds for W_1.
- We see that W_1, W_2 and W_2^T are not isomorphic to each other.
- All the other matrices $W(n, 16), n = 25, 27, 29$ found by the tiling method satisfied that $MCI(W, 3) \neq MCI(W^T, 3)$.
- However, as it will turn out, $W_1 \simeq W_1^T$.

MCI as an (strong?) invariant

- $MCI(,d)$ is a Hadamard Equivalence invariant which is
  - Computable.
  - Quite strong.
- $MCI(,3)$ is not strong enough to separate all isomorphism classes. Examples:
  - $W(13,9)$ has 8 isomorphism classes. $MCI(,3)$ can only distinguish 7.
  - $MCI(,3)$ is constant along *all* Hadamard matrices of size $4n$.
- Using Craigen's Weaving technique, for every d it is possible to construct examples in which $MCI(,d)$ is not completely separating.
- Nevertheless, it can give us a lot of information and help us to construct isomorphisms when they exist.

Extendable Partial Isomorphisms

Definition

Let A and B be $m \times n$ $\{0, -1, 1\}$-matrices. A partial isomorphism is an isomorphism between SA and SB, two partial $d \times n$ submatrices of A and B.

- When $MCI(A, d) = MCI(B, d)$, this gives rise to many partial isomorphisms from A to B.
- Question: How do we know whether a partial isomorphism extends to a full isomorphism $A \simeq B$?

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- Question: How do we know whether a partial isomorphism extends to a full isomorphism $A \simeq B$?
- Answer:(For square nonsingular A) If it extends, then in $L_1 \cdot SA \cdot R_1^T = SB$ and its extension $LAR^T = B$, we must have $R_1 = R$.
- Therefore, we can extract L:

$$L = BRA^{-1} = BR_1A^{-1},$$

and test that L is indeed monomial.

From MCI to the full isomorphism space $Isom(A, B)$-First method

Here's an algorithm:

1. Suppose that A, B are two nonsingular, square matrices, with $MCI(A, d) = MCI(B, d)$.
2. Take two isomorphic $d \times n$ matrices SA, SB.
3. Enumerate over all isomorphisms $SA \to SB$ and test if they are extendable.
4. Repeat for the same SA and all other SB having the same CI.

Complexity issues

- It is desirable to choose SA with the least multiplicity in $MCI(A)$.
- The number of isomorphisms $SA \to SB$ may be too large because:
  - SA (and SB) may have many zero columns, and there is a cost factor of 2 per each.
  - SA (and SB) may have too many repeated columns, and there is a factorial cost factor per each.
- These problems happen when d is small, and greatly reduce as d grows.
- On the other hand there is a cost of $2^d d!$ due to the unknown L_1 (and the computation of MCI).
- Considering only matrices with $d+1$ rows that include the given $S(A)$ as a submatrix reduce the cost of increasing $MCI(A,d) \to MCI(A,d+1)$.

Complexity issues

- It is hard to estimate complexity.
- Worst case complexity is quite large: For $A = B =$ identity matrix, there are $2^n n!$ isomorphisms, so it is at least as bad as $\Omega(2^n n!)$.
- But for 'heavy' weighing matrices $A = W(n, w)$ with $n/w = O(1)$, the 'average' case complexity is quite reasonable:
  - Repetitions and zero colums in SA stop in probability as soon as $d = O(\log n)$. The MCI cost is then $2^d d! \binom{n}{d} dn^2 \log n = O((2n)^{d+4})$. Hence the 'average' case complexity is at most

$$Av = (2n)^{O(\log n)}.$$

Our experience with $W(23, 16)$

Let W be the $W(23, 16)$ found by the Shaddow Geometry method. We wanted to compute $Isom(W, W^T)$.

- Starting with $MCI(W, 3)$ and choosing a specific SA, we saw multiple columns with multiplicities $[5, 4, 4, 4, 3]$ and the 3 repeating columns were zero columns. This implies that

$$\#Isom(SA, SB) \geq 5!4!^3 3!2^3 = 39813120,$$

  which is quite big.
- Then we went to $MCI(W, 4)$. We multiplicities of columns in SA were

$$[4, 3, 2, 2, 2, 2, 2, 2,]$$

  and the 3 repeating columns were zero columns Hence

$$\#Isom(SA, SB) \geq 4!3!(2!)^6 2^3 = 73728.$$

  Much better.

Our experience with $W(23,16)$

- Then, in $MCI(W,5)$ the multiplicities in SA reduced to $[2,2,2,3,3]$ with no zero column. Therefore we got an estimate of

  $$\#Isom(SA,SB) \geq 2!^3 3!^2 = 288.$$

  This turned out to be the best setting, regarding the cost of the MCI.
- At the bottom line, we found that $Isom(W,W^T)$ contains 6 elements.

From MCI to the full isomorphism space *Isom*(A, B)-Second method

- The algorithm presented above to retrieve a full isomorphism has some difficulties. There is an alternative method.
- We call the first method the inversion method and the second one the eigenmethod.
- The eigenmethod
  - works for A and B which are not necessarily square and regular.
  - Reduces to the case of true permutation isomorphisms.
  - Works with smaller d.
  - has less complexity of running time.
  - has less complexity of conditions to assume prior to the implementation
  - but has a more complicated algorithm.
- We will discuss the eigenmethod only briefly without supporting examples.

The eigenmethod

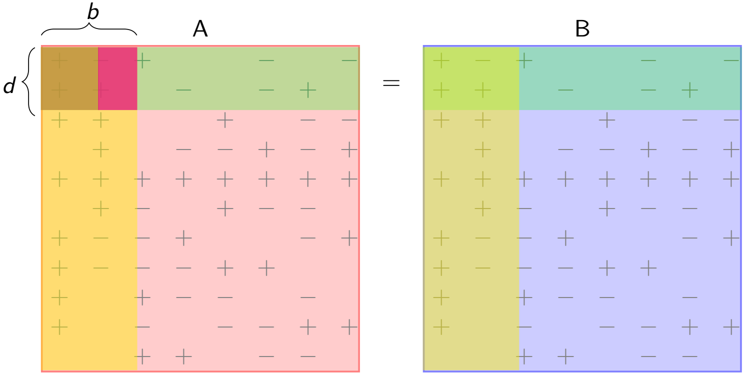
- We begin with a partial isomorphism $SA \to SB$, $L_i(SA)R_i^T = SB$.
- WLOG: Assume that SA=SB are positioned at the top layer of A and B and are of canonical form.

Outline

- I. Modify A by signs in rows and columns, so that $B = PAQ^T$ with permutation matrices P and Q.
  II. Discover P by the *eigenvalue method*.
  III. Discover Q by column sorting.

Stage I: The eigenmethod - Getting rid of signs

- The fact that $SA = SB$ still leaves many candidate isomorphisms $SA \to SB$ to be extended to $A \to B$.
- We divide the matrices A and B as



- A similar partitiom of A was used constructing the shaddow geometry [G].

Stage I: The eigenmethod - Getting rid of signs

- Assume: The first b columns of SA are different from the last $n - b$ columns.
- The algorithm:
    - Permute equal columns only in the corner of A.
    - Normalize the rows of A starting from row $d + 1$.
    - Enumerate on the signs of columns and rows which are zero within the margins of A.
- Under the enumeration, compute the *Can* of the vertical parts of the margins of A and B and test if they are equal.
- If equal, we may assume that $PAQ^T = B$ for permutation matrices P and Q. Continue to the next stage.

Stage II: Discovering P by eigenvectors

- If $PAQ^T = B$, then $PAA^TP^T = BB^T$, so AA^T and BB^T are similar by P.
- For Weighing Matrices $AA^T = BB^T = I_m$ there is no information.
  - We may apply a pointwise functions f so $Pf(A)Q^T = f(B)$.
  - Also transpose inversion works: $Pf(A)^{-1T}Q^T = f(B)^{-1T}$.
  - May combine few such operations and finally take Gram products.
- We end up with matrices G_A and G_B similar by P:

$$PG_AP^T = G_B.$$

At this stage we may test whether they have the same characteristic polynomial.

Stage II: Discovering P by eigenvectors

- P transforms the eigenspaces of G_A to those of G_B.
- Simplest Case: For an eigenvalue λ, the eigenspace is simple 1 dimensional, and the eigenvector $v_{\lambda,A}$ of A has distinct entries.
- In this case, we readily read the permutation P from

$$Pv_{\lambda,A} = v_{\lambda,B}.$$

- When this does not happen, it is mainly because there are many solutions for P. There are good ways to solve this, but we will not explore them here.

Stage III: Finding Q

- If P is known, we still need to find Q.
- Suppose that $PAQ^T = B$ with PA known. Then
  - If A is invertible, $Q = B^{-1}PA$.
  - In General: Find out Q by sorting the columns of PA and B.

An (anti)symmetric representative in a class

- Suppose that there exists a (anti)-symmetric S in the isomorphism class of A.
- Exercise: There exists an isomorphism $LAR^T = A^T$, such that $L = \pm R^T$.
- So going over all elements of $Isom(A, A^T)$, we can find such an isomorphism.
- Now, $LA = A^T R = (R^T A)^T = (\pm LA)^T$ is an (anti)symmetric matrix Hadamard equivalent to A.
- We found one symmetric representative $W(23, 16)$ for the $W(23, 16)$ found by the shaddow geometry method and 12 symmetric representatives for $W(7, 4)$ which are not equivalent by means of Symmetric Hadamard equivalence.

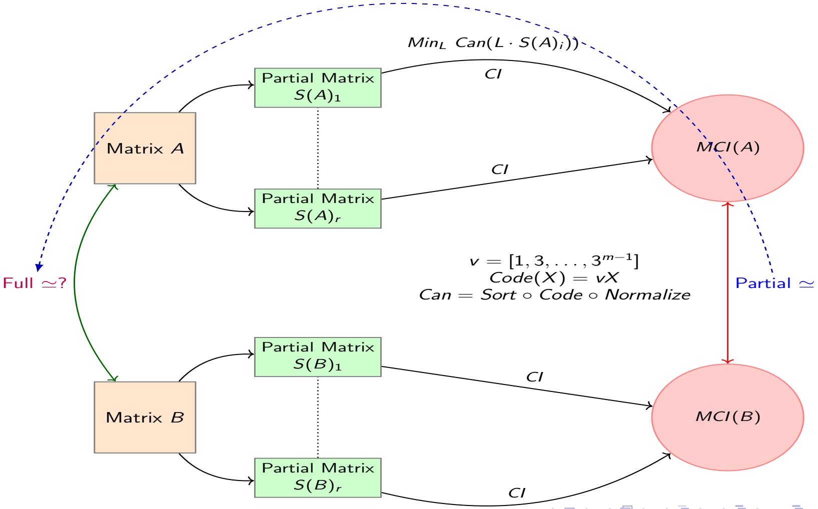
An example of a symmetric $W(7,4)$ obtained for the given one by our method

$$W = \begin{pmatrix} 0 & 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 & -1 & 0 & -1 \\ 1 & 0 & -1 & 0 & -1 & 0 & 1 \\ -1 & 1 & 0 & 0 & -1 & -1 & 0 \\ -1 & -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & -1 & -1 \\ 0 & -1 & 1 & 0 & 0 & -1 & 1 \end{pmatrix}$$

An example of a symmetric $W(23, 16)$ obtained for the given one by our method

$$
\left(\begin{array}{rrrrrrrrrrrrrrrrrrrrrrr}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 \\
0 & 0 & 0 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \\
0 & 0 & 0 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & -1 & -1 & 0 & -1 & -1 & -1 & -1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & -1 & 0 & 0 & 1 & -1 & -1 \\
1 & 1 & -1 & 1 & 0 & 1 & 1 & -1 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & -1 & 1 \\
1 & 1 & -1 & 1 & 0 & -1 & 1 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 1 & -1 & 0 & 1 & 1 & 1 & -1 & 0 & -1 \\
1 & 1 & -1 & -1 & 0 & 1 & -1 & 1 & 0 & -1 & -1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & -1 & -1 & -1 & 1 & 0 \\
1 & 1 & 1 & 0 & -1 & 1 & 1 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & 1 & 1 & 0 & -1 & -1 & 1 & 0 & 1 & -1 \\
1 & 1 & 1 & 0 & 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 & -1 & -1 & 1 & 0 & -1 & 1 & 1 \\
1 & 1 & 1 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & 0 & -1 & -1 & 1 & 0 & 1 & 1 & -1 & 0 \\
1 & 1 & 1 & 0 & -1 & -1 & 1 & 0 & 0 & 1 & 0 & 1 & -1 & 0 & -1 & -1 & 1 & 0 & -1 & -1 & 1 & 0 & 1 \\
1 & -1 & -1 & 1 & -1 & 0 & -1 & 1 & 1 & -1 & 0 & 1 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 1 & 1 & 0 & -1 \\
1 & -1 & -1 & -1 & 1 & 0 & 1 & 1 & -1 & 0 & 1 & 0 & -1 & -1 & 1 & 0 & -1 & 0 & 0 & -1 & 1 & 1 & 0 \\
1 & -1 & -1 & 1 & -1 & 0 & -1 & -1 & 0 & 1 & 1 & -1 & -1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 1 & 1 \\
1 & -1 & -1 & -1 & 1 & 0 & 1 & 0 & 1 & 1 & -1 & -1 & 1 & 0 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & -1 & 1 \\
1 & -1 & 1 & -1 & -1 & 1 & 0 & 0 & -1 & 1 & -1 & 1 & 0 & 1 & -1 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & -1 \\
1 & -1 & 1 & 1 & 1 & -1 & 0 & 1 & 0 & -1 & -1 & 0 & -1 & 1 & 1 & 1 & 1 & -1 & 0 & 0 & 0 & -1 & 0 \\
1 & -1 & 1 & 1 & 1 & 1 & 0 & -1 & -1 & 0 & 1 & 1 & 1 & -1 & 0 & 0 & 1 & 1 & -1 & 0 & -1 & 0 & 0 \\
1 & -1 & 1 & -1 & -1 & -1 & 0 & -1 & 1 & -1 & 0 & -1 & 1 & 0 & 1 & -1 & 0 & 1 & 1 & -1 & 0 & 0 & 0
\end{array}\right)
$$

A Schematic Diagram



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- The site of Jeff Dinitz -content of the second edition of the book Handbook of Combinatorial Designs edited by Chales J Colbourn and by Jefferry H. Dinitz the table of content http://www.emba.uvm.edu/ jdinitz/contents.html. New weighing matrices http://www.emba.uvm.edu/ jdinitz/part5.newresults.html

Bibliography

The site of Akihiro Munemassa
http://www.math.is.tohoku.ac.jp/ munemasa/index-e.html
has a link to new unpublished weighing matrices
http://www.math.is.tohoku.ac.jp/ munemasa/research/matrices/wo.htm