Constructions of matrices from the developed cohomology filtration

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Galway May 23rd 2019

Overview

Weighing matrices

- A weighing matrix $W(N, w)$ is a $N \times N$ matrix W whose elements are $0, \pm 1$ such that $WW^T = wI_N$. $W(N, w)$ denotes both a single matrix and the class of all $W(N, w)$.
- The following are $W(2, w), 1 \leq w \leq 2$

$$\left(I_2 \;\middle|\; \begin{matrix} 1 & 1 \\ 1 & - \end{matrix} \right)$$

- The following are $W(3, w), 1 \leq w \leq 3$

$$\left(I_3 \;\middle|\; None \;\middle|\; None \right)$$

- The following are $W(4, w), 1 \leq w \leq 4$

$$\left(\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix} \;\middle|\; \begin{matrix} 1 & 1 & 0 & 0 \\ 1 & - & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & - \end{matrix} \;\middle|\; \begin{matrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & - \\ 1 & - & 0 & 1 \\ - & - & 1 & 0 \end{matrix} \;\middle|\; \begin{matrix} 1 & 1 & 1 & 1 \\ 1 & - & 1 & - \\ 1 & 1 & - & - \\ 1 & - & - & 1 \end{matrix} \right)$$

Weighing matrices

- Applications: Chemistry, Spectroscopy, Quantum Computing and Coding Theory.
- Main question: Determine the parameters for which a $W(N, w)$ exists.
- The Hadamard Conjecture: $H(N) = W(N, N)$ exists for every $N = 4k, k \in \mathbb{N}$.
- To date the smallest unkonwn Hadamard matrix is $H(668)$.
- The previous unknown Hadamard matrix $H(428)$ was found by H. Kharaghania,b,. B. Tayfeh-Rezaiea in the year 2005
- Some well-known families of Hadamard and weighing matrices. Let q be an odd prime power.
  - Payley Conference Matrices: $W(q+1, q)$
  - Payley Hadamard Matrices: $H(N)$ for $N = q+1$ or $2q+2$.
  - Projective Space Matrices: $W(\frac{q^{d+1}-1}{q-1}, q^d)$.

Generalized Weighing Matrices

More generally, let

$$\mu_n = \text{ all roots of unity pf order } n, \ \mu_n^+ = \mu_n \cup \{0\}.$$

Definition

A Generalized Weighing Matrix is a μ_n^+-matrix W such that

$$WW^* = wI_N, \ (\text{*=conjugate-transpose}) \ .$$

- Even more generally: We may consider matrices with values in $\mu^+ = \mu \cup \{0\}$, where $\mu \subseteq R^\times$ for a commutative ring R.
- We define μ^+-valued weighing matrices similarly, with taking, for $W_{i,j} \in \mu$, $W_{i,j}^* := W_{j,i}^{-1}$.

Cocyclic Matrices

Definition (Group Development)

Let G be a finite group. A matrix A indexed by G is G-developed if it has the form

$$A_{g,h} = f(gh).$$

- It is well known that a G-developed weighing matrix, must have weight $w = l^2$.
- A modification of this is the notion of a Cocyclic Matrix (Horadam, DeLauney, Flannary, Egan,. . . [3])

Definition (Cocyclic Matrices)

A cocyclic G-matrix is a matrix indexed by G of the form

$$A_{g,h} = \omega(g,h)f(gh),$$

where $\omega : G \times G \to \mathbb{C}^{\times}$ is a 2-cocycle.

An example of a G develpoed matrix

- Let $G = Z_6, X = Y = \{1,2,3,4,5,6\}$, the set of vertices of the hextagon. G- acts trivially on μ.
- The matrix is constant along the G-orbits, and spanned by its first row. Plugging 1st row $(-,+,0,+,+,0)$ gives $W(6,4,2)$.

$$\begin{pmatrix} o1 & o2 & o3 & o4 & o5 & o6 \\ o2 & o3 & o4 & o5 & o6 & o1 \\ o3 & o4 & o5 & o6 & o1 & o2 \\ o4 & o5 & o6 & o1 & o2 & o3 \\ o5 & o6 & o1 & o2 & o3 & o4 \\ o6 & o1 & o2 & o3 & o4 & o5 \end{pmatrix}, \begin{pmatrix} -1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 & -1 & 1 \\ 1 & 1 & 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 & 1 & 1 \end{pmatrix}$$

An example of a cocyclic matrix

- For the same X, Y, G set $\mu = \{\pm 1\}, \chi : Z_6 \times Z_6 \mapsto \mu$ by $\chi(x,y) = 1$ for $x + y < 6$ $\chi(x,y) = -1$ for $x + y \geq 6$, where the addition is taken modulu Z.
- χ satisfies the coboundary formula $\chi(y,z) - \chi(x+y,z) + \chi(x,y+z) - \chi(x,y)$
- The same 6 orbits change so that each term is negated moving from the upper to the lower part of the backtriangular .

$$\begin{pmatrix} o1 & o2 & o3 & o4 & o5 & o6 \\ o2 & o3 & o4 & o5 & o6 & -o1 \\ o3 & o4 & o5 & o6 & -o1 & -o2 \\ o4 & o5 & o6 & -o1 & -o2 & -o3 \\ o5 & o6 & -o1 & -o2 & -o3 & -o4 \\ o6 & -o1 & -o2 & -o3 & -o4 & -o5 \end{pmatrix}$$

An example of a cocyclic matrix

- Plugging 1st row $(-,-,-,0,+,-)$ gives $W(6,5,2)$

$$\begin{pmatrix} -1 & -1 & -1 & 0 & 1 & -1 \\ -1 & -1 & 0 & 1 & -1 & 1 \\ -1 & 0 & 1 & -1 & 1 & 1 \\ 0 & 1 & -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & 0 \\ -1 & 1 & 1 & 1 & 0 & -1 \end{pmatrix}$$

- The last two examples are of back-cyclic and back-nega-cyclic matrices.

Our Work

- We present a theory that generalizes Cocyclic Matrices.
- Our work has relations to:
  - Automorphisms and Hadamard Operations
  - Hecke Algebras
  - Association Schemes
  - Brauer Groups
  - Higher Tensors and Designs - Trigebras
  - Magic Squares
  - Spectral Sequences.

Hadamard Operations and Automprphisms

- Let $Maps(M, N, \mu)$ be the space of all $M \times N$ μ^+-matrices.

Definition

- A Hadamard operation on $Maps(M, N, \mu)$ is given by multiplying both sides with monomial matrices, specifically by
  1. Axis permutations,
  2. Row or column multiplication by signs in μ,
  3. All compositions thereof.
- Denote the group of all Hadamad operations on $Maps(M, N, \mu)$ by $Had(M, N, \mu)$.
- Let $Aut(A)$ be the subgroup of $Had(M, N, \mu)$ preserving A.

Definition

Given $A, B \in Maps(M, N, \mu)$, the Hadamard Product $A \circ B$ is defined by

$$(A \circ B)_{i,j} = A_{i,j} B_{i,j}.$$

Hadamard equivalence of Weighing Matrices

- A monomial matrix (a signed permutation) P is a permutation matrix whose non zero elements are in μ.
- Two matrices U, V are isomorphic (=Hadamard equivalent) if there exists two monomial matrices P, Q with $PUQ^* = V$.
- The following exhibits an Hadamard equivalence between the Sylvester matrix $H2 \otimes H2$ and the circulant Hadamard matrix $CH4$.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & - \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & - & 1 & - \\ 1 & 1 & - & - \\ 1 & - & - & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ - & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} - & 1 & 1 & 1 \\ 1 & - & 1 & 1 \\ 1 & 1 & - & 1 \\ 1 & 1 & 1 & - \end{bmatrix}$$

Well known classifications of Hadamard equivalence classes

- the present statements concern classical matrices $\mu = \{\pm 1\}$.
- There is one class for Hadamrad matrices of each order $N \leq 12$, 5 for $N = 16$, 3 for $N = 20$, and 60 for $N = 24$.
- Chan Rodger Seberry [CRS] classified all $W(N, w)$ for $w \leq 5$ and all $W(N, w)$ for $N \leq 11$.
- Harada and Munemasa [HM] classified all $W(N, w)$ of $N \leq 15$, $N = 17$, $W(16, w), w = 6, 9, 12$, and $W(18, 9)$. They found 11891 classes of $W(18, 9)$.

Geometries and Automorphisms

Definition

A (μ_n) support geometry is an w-biregular $N \times N$ $\{0,1\} = (\mu_1^+)$-matrix S such that

$$SS^T \equiv wI_N \mod n.$$

(Chan-Rodgers-Seberry [CRS])

- If W is a weighing matrix, then $|W|$ is a support geometry.
- We define the shadow geometry of W as $J - |W|$ where J is the all-1 matrix. [1].
- Hadamard operations on W reduce to permutations on $|W|$. Hence there is a homomorphism

$$Aut(W) \to Aut(|W|)$$

.
- In general this homomorphism is neither injective or surjective.

The Automorphism Lifting Problem (ALP)

- One possible way to find weighing matrices $W(N, w, n)$, is to
  1. Find a support geometry S with parameters N, w, n
  2. "Color" it by signs.
- Automorphisms may guide us in how to "Color".

The Automorhism Lifting Problem (ALP)

Given a subgroup $G \leq Aut(|W|)$, can we find a lifting W and $\hat{G} \leq Aut(W)$, such that $\hat{G} \to G$ is onto? .

- In general we do not expect the lifting to be a weighing matrix, but sometimes it will be.
- This has been fruitful in some cases (e.g. $W(23, 16, 2)$[4]).

A well known example

- This is the well-known weighing matrix

$$W = W(7,4) = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & - & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & - & 0 & - & 0 & 1 \\ 1 & 0 & 0 & - & 0 & - & - \\ 0 & 1 & - & 0 & 0 & 1 & - \\ 0 & 1 & 0 & - & 1 & 0 & 1 \\ 0 & 0 & 1 & - & - & 1 & 0 \end{pmatrix}$$

- Its shadow geometry is the incidence matrix of the Fano Plane.
- It can be shown that $Aut(W) \to Aut(|W|)$ is an isomorphism.
- So W is a solution to the ALP, for the full group of automorphisms $G = Aut(|W|)$.
- How general is this phenomena?

The Grassmannian Weighing Matrix Family

We present the following Grassmannian family which satisfes that

$$Aut(W) \twoheadrightarrow Aut(|W|) \text{ is surjective.}$$

Theorem (Grassmannian Weighing Matrices)

For every prime-power q, and integers $0 < k < d$ and $1 < n|q-1$, there is a generalized weighing matrix

$$GW\left(\begin{bmatrix} d \\ k \end{bmatrix}_q, q^{k(d-k)}, n\right), \quad Aut(W) \twoheadrightarrow Aut(|W|).$$

- $\begin{bmatrix} d \\ k \end{bmatrix}_q$ are the gaussian binomial coefficients.
- plugging $d+1$ in d and 1 in k gives the classical projective space matrices.

Flag Variety Weighing Matrices

The following family reduces in the case $r = 1$ to the previous one:

Theorem (Flag Variety Weighing Matrices)

Let q be a prime-power, let $0 < k_0 \leq k_1 \leq \cdots \leq k_r$ be integers, let $d = \sum k_i$, and let $n|q-1$. Assume that $n \geq r$. Then there exists a generalized weighing matrix,

$$GW\left(\begin{bmatrix} d \\ k_0, k_1, \ldots, k_r \end{bmatrix}_q, q^{\sum_{i<j} k_i k_j}, n\right), \quad Aut(W) \twoheadrightarrow Aut(|W|).$$

Comments

Unlike Grasmannians and Projective Spaces, if F is a Flag matrix, and $m|n$, then $F^{\circ(n/m)}$ may not be a weighing matrix. Hence F is not balanced.

Quasi Projective/Grassmannian/Flag Weighing Matrices

Theorem (Quasi Projective/Grassmannian/Flag Weighing Matrices)

Assume that the conditions of the previous theorem are satisfied and $GW(N, w, n)$ is the matrix guaranteed by the theorem. Suppose that a circulant weighing matrix $CW(M, t, n)$ exists, for some $M|q-1$. Then there can be constructed a generalized

$$W = GW(MN, tw, n), \quad Aut(W) \twoheadrightarrow Aut(|W|),$$

and in general $W \not\simeq GW(N, n, n) \otimes CW(M, t, n)$.

Remarks:

1. This construction is sort of a Twisted Kronnecker Product.
2. W cannot be obtained from both matrices by Craigen's weaving technique, as can be shown by computing a certain invariant.

Three families of Hadamard matrices

Theorem

for every prime power $q \equiv 1 \mod 4$ *, there exists an Hadamard marix* $H(4q(q+1))$.

Theorem

for every prime power $q \equiv 3 \mod 4$, *such that* $q-4$ *is a prime power too , there exists an Hadamard marix* $H(8q(q-3))$.

Theorem

for every prime power $q \equiv 3 \mod 4$, *such that* $q-2$ *is a prime power too , there exists an Hadamard marix* $H(8q(q-1))$.

Remarks on the ALP

Special Remarks

- All families described above were found by solving instances of the ALP.
- The (Quasi) Projective/Grassmannian/Flag matrices are ALP lifts of incidence matrices (Support Geometries) of the (Quasi) Projective/Grassmannian/Flag over finite fields.
- The Hadamard families, were built from rectangular blocks, each of which was obtained as an instance of an ALP.

General Remarks

- The ALP has the trivial solution $W = |W|$, $\hat{G} = G$, but this solution can not produce weighing matrices.
- We present a (feasible) computational algorithm to solve the ALP, but no algorithm to guarantee that W is a $W(N, w, n)$.
- Our algorithm uses the cohomology groups $H^i(G, -), i = 1, 2$.

Modified Hadamard Operations

- Suppose that we have a twist action on μ (e.g. $\zeta \mapsto \zeta^a$).

Definition

- A Modified Hadamard operation on $Maps(M, N, \mu)$ is by
  1. Axis permutations,
  2. Row or column multiplication by signs in μ,,
  3. Twisting the coefficients.
  4. By all compositions thereof.
- Denote (by abuse) the group of all Modified Hadamard operations on $Maps(M, N, \mu)$ by $Had(M, N, \mu)$.
- Let $Aut(A)$ be the subgroup of $Had(M, N, \mu)$ preserving A.

Basic Setting

We choose:

1. A finite group G.
2. Two finite G-sets, X, Y.
3. An (multiplicative) abelian group μ equipped with G-action.
4. A G-stable subset $\mathcal{O} \subseteq X \times Y$.

$\mathcal{O}$-matrices

- An $\mathcal{O}$-matrix is a matrix $A = (f(x,y))_{X \times Y}$ such that $f(x,y) \in \mu^+ := \mu \cup \{0\}$, and $f(x,y) \neq 0$ implies $(x,y) \in \mathcal{O}$.
- The set of $\mathcal{O}$ matrices is denoted $Maps(\mathcal{O}, \mu)$.
- $g \in G$-acts on $A \in Maps(\mathcal{O}, \mu)$-by

$$(gA)_{x,y} = gA_{g^{-1}x, g^{-1}y}.$$

Cohomology Developed Matrices

Two $\mathcal{O}$-matrices A, B are D-equivalent if there exist diagonal μ matrices L, R such that

$$A \sim_D B \Leftrightarrow A = LBR^*, .$$

Definition

An $\mathcal{O}$-matrix is Cohomology Developed if

$$gA \sim_D A, \quad \forall g \in G.$$

The set of all cohomology develped $\mathcal{O}$-matrices (CDM's) denoted $\mathfrak{Coh}(\mathcal{O}, \mu)$ is a subgroup of $Maps(\mathcal{O}, \mu)$ (w.r.t the Hadamard (=componentwise) product).

G-Developed and Cocyclic Matrices are cohomology developed

- Suppose that $X = G$ with action $x \mapsto xg^{-1}$ and $Y = G$ with action $y \mapsto gy$. Then a G-developed matrix satisfies

$$gA = A \; \forall g \in G.$$

- By an easy calculation, all G-Cocyclic Matrices are Cohomology Developed:

$$\begin{aligned}\omega(xg^{-1}, gy)f(xg^{-1}gy) &= \omega(g^{-1}, gy)\omega(x, y)f(xy)\omega(x, g^{-1})^{-1} \\ &\sim_D \omega(x, y)f(xy).\end{aligned}$$

More examples

Let $G = D_5$, and let $X = Y = \{1,2,3,4,5\}$, the set of vertices of the pentagon. G- acts trivially on μ.

Let

$$A = \begin{pmatrix} x_0 & x_1 & x_2 & x_2 & x_1 \\ x_1 & x_0 & x_1 & x_2 & x_2 \\ x_2 & x_1 & x_0 & x_1 & x_2 \\ x_2 & x_2 & x_1 & x_0 & x_1 \\ x_1 & x_2 & x_2 & x_1 & x_0 \end{pmatrix}$$

Then

$$\forall g \in G \quad gA = A.$$

This is an example of a modified G-developed matrix.

More examples

Let $G = \langle \sigma \rangle$, $\sigma^2 = 1$, $X = Y = G$, and let $\mu = \mathbb{C}^\times$, where σ acts by complex conjugation.

Let

$$A = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}.$$

Then

$$\sigma A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} A \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}^* .$$

This is an example of a modified G-cocyclic matrix.

More Examples - The Fourier Matrix

Let $X = Y = \mathbb{Z}/n$,

$$G = \{x \mapsto ax + b \mid a \in (\mathbb{Z}/n)^{\times},\ b \in \mathbb{Z}/n\}$$

be the Affine group. Let $\mu = \mu_n = \langle \omega \rangle$, $\omega = exp(2\pi i/n)$, and $g = (x \mapsto ax + b)$ acts on μ by $g\omega = \omega^{a^2}$.

Define the Fourier matrix

$$F = (\omega^{xy}).$$

Then one can check that

$$gF = LFR^*,$$

where $L = R^* = diag(\omega^{-bx}\omega^{-b^2/2})$.

More examples - Rectangular matrices

Let $G = B_3$ be the symmetry group of the cube. Let

- $X = Faces$,
- $Y = Vertices$
- $Z = Edges$.
- $\mu = \{\pm 1\}$

Then

$$A = \begin{pmatrix} -1 & 0 & -1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 1 & -1 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ on } X \times Y,$$

and

$$B = \begin{pmatrix} -1 & 1 & 0 & 1 & -1 & 0 & -1 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & -1 & -1 & 1 & -1 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & -1 & -1 & -1 & 1 & -1 \\ 1 & -1 & 0 & -1 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & 1 & 0 & -1 & 0 & -1 & 0 & 0 \\ -1 & 1 & 1 & 1 & -1 & 1 & 0 & -1 & 0 & -1 & 0 & 0 \end{pmatrix} \text{ on } X \times Z$$

are cohomology developed.

The groups $\widetilde{Had}$, Had and a cover $re : \widetilde{Had} \twoheadrightarrow Had$

- Given sets X, Y, and an abelian group μ, define the set $\widetilde{Had(X, Y, \mu)} = \{(P, Q, \lambda), \}$ where $P(Q)$ is a monomial matrix indexed by $X(Y)$, respectively, and λ is an automorphism of μ.
- $\widetilde{Had}$ acts on $Maps(|X|, |Y|, \mu)$ by $(P, Q, \lambda)A = PA^{\lambda}Q^{*}$ where A^{λ} is obtained by acting with λ componentwise.
- $\widetilde{Had}$ has an induced transformation group structure with $(P, Q, \lambda) \circ (R, S, \tau) = (PR^{\lambda}, QS^{\lambda}, \lambda \circ \tau)$.
- There is a realization map $re : \widetilde{Had} \mapsto Had$ sending (P, Q, λ) to the homomorphism $A \mapsto PA^{\lambda}Q^{*}$.
- re is an epimorphism of groups but **not** 1-1.
- $ker(re)$ is isomorphic to μ and sends $\nu \in \mu$ to $(\nu I_X, \nu I_Y, Id)$.

The groups $Had(\mathcal{O})^{\sim}, Had(\mathcal{O})$ and a cover $re : Had(\mathcal{O})^{\sim} \twoheadrightarrow Had(\mathcal{O})$

- Given sets X, Y, and a subset $\mathcal{O} \subseteq X \times Y$ define p_x (p_Y) to be the projection of $\mathcal{O}$ onto the X (Y) axis respectively.
- Given also an abelian group μ define $Had(\mathcal{O})^{\sim}$ to be $Had(p_X, p_Y.\mu)^{\sim}$, In particular if $p_X = X, p_= Y$ then $Had(\mathcal{O})^{\sim} = Had(X, Y, \mu)^{\sim}$.
- Let $Had(\mathcal{O})$ be the group of all Hadamard operations on $Maps(\mathcal{O}, \mu)$. In particular there is the cover map $re : Had(\mathcal{O})^{\sim} \twoheadrightarrow Had(\mathcal{O})$.

The fiber of *re*

- $\mathcal{O} \subseteq X \times Y$ is called reducible if there exist non empty partitions X_1, X_2 of X and Y_1, Y_2 of Y such that $\mathcal{O} \subseteq (X_1 \times Y_1) \cup (X_2 \times Y_2)$. Otherwise $\mathcal{O}$ is irreducible.
- The reducibility degree $rd(\mathcal{O})$ of $\mathcal{O}$ is the largest K so that there exist partitions of X (Y) with K subsets X_i (Y_i) respectively so that $\mathcal{O} \subseteq \cup_{i=1,K}(X_i \times Y_i)$.
- The reducibility degree of $\mathcal{O} = X \times Y$ is 1.
- The fiber of $re \cong \mu^{\times rd(\mathcal{O})}$, where $\nu \in \mu_i$, the i^{th} factor of μ, multiplies the coordinates in X_i and Y_i by ν.

A commutative diagram

Given $A : X \times Y \mapsto \mu$, $Aut(A)^{\sim}$ $(Aut(A))$ denotes its stablizer in $Had^{\sim}$ (Had) and given $(A|\mathcal{O}) : \mathcal{O} \mapsto \mu$, $Aut(A|\mathcal{O})^{\sim}$ $(Aut(A|\mathcal{O}))$ denotes its stablizer in $Had(\mathcal{O})^{\sim}$ $(Had(\mathcal{O}))$ respectively.
In the category of groups, there is a commutative diagram:

$$
\begin{array}{ccccccc}
 & & \mu^{\times rd(\mathcal{O})} & \xhookrightarrow{=} & \mu^{\times rd(\mathcal{O})} & \xleftarrow{diag} & \mu \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & Aut(A|\mathcal{O})^{\sim} & \xhookrightarrow{def^{\sim}} & Had(\mathcal{O})^{\sim} & \xleftarrow{rest} & Had^{\sim} \\
G & \overset{?\rho?}{\dashrightarrow} & \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\
G & \underset{?\overline{\rho}?}{\dashrightarrow} & Aut(A|\mathcal{O}) & \xhookrightarrow{def} & Had(\mathcal{O}) & \xleftarrow{rest} & Had,
\end{array}
$$

where all columns are short exact sequences, the horizontal right (left) maps are induced by restrictions (inclusions) respectively.
The right column is defined for all sets X, Y and an abelian group μ, the middle column is defined given the above and $\mathcal{O} \subseteq X \times Y$ and the left assuming the above and $A|\mathcal{O} \in Maps(\mathcal{O}, \mu)$.

The G action

- If G acts on X, Y and on μ. Then the right column in the previous diagram lies in the category of G groups and maps.
- If further $\mathcal{O} \subseteq X \times Y$ is a stable G-set, then also the middle column lies in the G-category.
- If further $A|\mathcal{O} : X \times Y \mapsto \mu$ is chosen to be a G map then the entire diagram lies in the G category.
- A G developement of $A|\mathcal{O} : \mathcal{O} \mapsto \mu$ is the existence of a homomorphism $\overline{\rho} : G \mapsto Aut(A|\mathcal{O})$.
- Assuming G, X, Y and μ are finite, for each such $\overline{\rho}$, there are finitely many lifts ρ. There is no canonical ρ and any chosen ρ need not be a homomorphism.

Example-ρ is not a homomorphism

- Recall the example from before, that

$$A = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}.$$

- One can choose $def^{\sim} \circ \rho$ (denoted ρ)

$$\rho(1) = (I, I, id_{\mu})$$
$$\rho(\sigma) = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, conj \right)$$

- But then

$$\rho(\sigma)^2 = (-I, -I, id_{\mu}) \neq \rho(1),$$

hence ρ isn't a homomorphism.

Construction of CDM's

From now on we abuse notations and denote $def^{\sim} \circ \rho$ ($def \circ \overline{\rho}$) by ρ ($\overline{\rho}$) respectively.

Given $G, X, Y, \mathcal{O}$, the construction has two stages:

1. Coloring: Construct a map $\rho : G \to Had(\mathcal{O})^{\sim}$ s.t. $\bar{\rho} : G \to Had(\mathcal{O})$ is a homomorphism.
2. Spanning Choose a value at a basepoint in each orbit of $\mathcal{O}$, and span the matrix by the G-action.

A filtration on CDM's

- Let $\mathfrak{Coh}(\mathcal{O},\mu)$= The set of all $\mathcal{O}$ CDM's.

Theorem

- *There is a filtration*

$$\mathfrak{Coh}^0(\mathcal{O},\mu) \subseteq \mathfrak{Coh}^1(\mathcal{O},\mu) \subseteq \mathfrak{Coh}^2(\mathcal{O},\mu) = \mathfrak{Coh}(\mathcal{O},\mu)$$

  of groups with respect to the Hadamard (=componentwise) product of matrices.
- *This filtration is closed under D-equivalence.*
- *The associated graded quotients relate to the cohomology of G.*

- $\mathfrak{Coh}^0(\mathcal{O},\mu)$ = the D-closure of the set of all G-invariant matrices.
- $\mathfrak{Coh}^1(\mathcal{O},\mu)$ = the set of all matrices, such that $\exists\rho : G \to Aut(A|\mathcal{O})\tilde{}$, a homomorpohism.

Spanning - Orientable Orbits

Assume that we have a coloring, that is, we know how G acts on the matrix.

A Problem

When spanning, different elements of G might give conflicting signs to the same matrix element! (Not to mention coefficient twists)

- When this happens to some (x, y), it happens to all points at its orbit.

In this case, we say that this orbit is non-orientable with respect to this coloring. Otherwise it is orientable.

- All values of points at non-orientable orbits must be set to 0.

Example $G = D_8$

- Let $G = D_8$, acting on $X = Y = \{0, 1, 2, 3\}$.
- Consider the characters $\chi, \mathbf{1} : G \to \mu_2$:

$$\chi = \text{ The determinant character,}$$

$$\mathbf{1} = \text{ The trivial character.}$$

- Let $\rho : G \to Had\tilde{}(X, Y)$ be the coloring

$$\rho(g) = (\chi(g)\Pi_X(g), \mathbf{1}(g)\Pi_Y(g)),$$

($\Pi(-)$=Permutation matrix).

Example $G = D_8$ Cont'd

- After applying $Z_4 \leq D_8$ the $X \times Y$-matrix has 4 orbits:

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 3 & 0 & 1 & 2 \\ 2 & 3 & 0 & 1 \\ 1 & 2 & 3 & 0 \end{pmatrix}.$$

- $\tau : (0,1,2,3) \mapsto (0,3,2,1) \in D_8 \setminus Z_4$ identifies orbits $0,1,2,3$ with orbits $0,3,2,1$ respectively, It also colors with multiplication by -1 any two identified orbits, giving the following colored matrix

$$\begin{pmatrix} \pm 0 & 1 & \pm 2 & -1 \\ -1 & \pm 0 & 1 & \pm 2 \\ \pm 2 & -1 & \pm 0 & 1 \\ 1 & \pm 2 & -1 & \pm 0 \end{pmatrix}.$$

- With respect to this coloring, orbits $0,2$ are non-orientable, and orbit 1 is orientable.

Example $G = B_3$

- Consider the symmetry group of the cube, B_3.
- We think of B_3 as the group of 3×3 monomial matrices.
- Let ($\mathbf{e}_i$ = the standard vectors)

$$F = \text{ Faces of the cube } = \{\pm\mathbf{e}_1, \pm\mathbf{e}_2, \pm\mathbf{e}_3\}$$

$$D = \text{ Antipodal Edges } = \{\pm\mathbf{e}_i \pm \mathbf{e}_j,\ i < j\}/\{\pm 1\}.$$

- There are two B_3-orbits in $F \times D$:

$$\begin{pmatrix} 1 & 2 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 & 2 \\ 2 & 1 & 1 & 2 & 1 & 1 \\ 2 & 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 & 2 \\ 1 & 2 & 2 & 1 & 1 & 1 \end{pmatrix}$$

Example - $G = B_3$ Cont'd

- There are independent characters $\chi_1, \chi_2 : B_3 \to \{\pm 1\}$.:

$$\chi_1(g) = \det(g)$$

$$\chi_2(g) = \text{ product of signs in } g.$$

- Let's look at the coloring $\rho_1 : B_3 \to Had(F, D)\tilde{}$:

$$\rho_1(g) = (\chi_2(g)\Pi_F(g), \chi_1(g)\Pi_D(g), id)\,.$$

  ($\Pi_X(-)$ is the permutation matrix).
- Then orbit 1 is orientable and orbit 2 isn't.

Example - $G = B_3$ Cont'd

- On the other hand, let $\rho_2 : B_3 \to Had(F, D)\tilde{}$:

$$\rho_2(g) = (\chi_1(g)\Pi_F(g), \chi_1\chi_2(g)\Pi_D(g), id)\,.$$

- This time it's the other way around: orbit 2 is orientable, and orbit 1 isn't.
- There are colorings with both orbits orientable, or none of them orientable.
- Both ρ_1, ρ_2 are homomorphisms, so the matrices are in $\mathfrak{Coh}^1$.
- The matrix for ρ_1 is:

$$\pm \begin{pmatrix} 1 & 0 & 0 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 & 1 \\ 0 & -1 & -1 & 0 & 1 & 1 \\ -1 & 1 & 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 1 & -1 & -1 \end{pmatrix}.$$

Coloring in $\mathfrak{Coh}^1$

- Matrices in $\mathfrak{Coh}^1$ are obtained from homomorphisms $\rho : G \to Had(\mathcal{O})\tilde{}$.
- Write $\rho(g) = (P(g), Q(g), \sigma(g))$. Then

$$P(gg') = P(g)P(g')^{\sigma(g)} \text{ and } Q(gg') = Q(g)Q(g')^{\sigma(g)}.$$

- Further, write $P(g) = L(g)\Pi_X(g)$ and $Q(g) = R(g)\Pi_Y(g)$. for diagonal L, R.

- We understand L, R as maps $L : X \to \mu$, and $R : Y \to \mu$.
- Turn $Maps(X, \mu)$ and $Maps(Y, \mu)$ to G-modules, where $gf(x) := g \cdot f(g^{-1}x)$.

Coloring in $\mathfrak{Coh}^1$

- We have

Lemma

$\rho : G \to Had(O)^{\sim}$ is a homomorphism, if and only if L, R are 1-cocycles:

$$L(gg') = L(g) + gL(g') \text{ and } R(gg') = R(g) + gR(g').$$

(written additively)

- The source of 1-cocycles is from cohomology. Then, we need to study the cohomology groups

$$H^1(G, M_X) \text{ and } H^1(G, M_Y),$$

where for brevity $M_X = Maps(X, \mu)$ and $M_Y = Maps(Y, \mu)$.

Coloring - Eckmann-Shapiro's Lemma

- M_X and M_Y turn out to be (co)-induced modules. In such cases we can invoke the Eckmann-Shapiro Lemma which states

Theorem (Eckmann-Shapiro Lemma)

Suppose that X, Y are G-transitive. Choose two basepoints $x_0 \in X$ and $y_0 \in Y$, and let $H_X = Stab(x_0)$ and $H_Y = Stab(y_0)$. Then there are isomorphisms

$$H^1(G, M_X) = H^1(H_X, \mu) \text{ and } H^1(G, M_Y) = H^1(H_Y, \mu).$$

Eckmann-Shapiro Lemma -Explicit form

- The isomorphism $H^1(G, M_X) = H^1(H_X, \mu)$ can be made explicit at the level of cocycles:

The map $H^1(G, M_X) \to H^1(H_X, \mu)$

Given a 1-cocycle $z : G \to M_X$, we map it to the 1-cocycle $y : H_X \to \mu$ by

$$y(h) = z(h)(x_0).$$

The map $H^1(H_X, \mu) \to H^1(G, M_X)$

For every $x \in X$, choose $g_x \in G$ s.t. $g_x x_0 = x$.
Given a 1-cocycle $y : H \to \mu$, we map it to the 1-cocycle $z : G \to M_X$, by

$$z(g)(x) = g_x \cdot y(g_x^{-1} g \; g_{g^{-1}x}).$$

A Criterion For Orientability

- We see that coloring in $\mathfrak{Coh}^1$ depends on the choice of two classes in $\psi_X \in H^1(H_X, \mu)$ and $\psi_Y \in H^1(H_Y, \mu)$.
- Let $O \subset \mathcal{O}$ be an orbit, and $(x, y) \in O$. The stabilizer of this point is

$$H_{x,y} = g_x^{-1} H_X g_x \cap g_y^{-1} H_Y g_y.$$

Hence there are embeddings $H_{x,y} \to H_X, H_Y$, and restriction maps

$$H^1(H_X, \mu) \xrightarrow{res_X} H^1(H_{x,y}, \mu) \xleftarrow{res_Y} H^1(H_Y, \mu).$$

Lemma

The orbit O is orientable, if and only if there is an equality of classes

$$res_X(\psi_X) = res_Y(\psi_Y).$$

Spanning of Orientable Orbits

- Let O be an orientable orbit.
- This means that there exists an element $\theta \in \mu$ from which we can span.
- In the presence of a twist, not every $\theta \in \mu$ is feasible. It needs to satisfy the equality of cochains

$$\psi_X(h)(h\theta)\psi_Y(h)^* = \theta \ \forall h \in H_{x,y}.$$

- The number of spans of O is the cardinality of $H^0(H_{x,y}, \mu)$.

Example-Projective Spaces

Fix

- A finite field $\mathbb{F}_q$,
- $\mu = \pm 1$,
- An even integer $d > 0$,
- $X = \mathbb{P}^d(\mathbb{F}_q)$, the Projective Space of dimension d,
- $G = PGL_{d+1}(\mathbb{F}_q)$, the group of Projective Transformations.

- G acts on X by matrix-vector multiplication, $x \mapsto gx$.
- Let $Y = X$, but with the G-action $y \mapsto {g^{-1}}^T y$.
- We think of X as the set of points, and Y as the set of hyperplanes.
- $X \times Y$ has 2 G-orbits:

$$O_0 = \{(x, y) | x^T y = 0\},$$

$$O_1 = \{(x, y) | x^T y \neq 0\}.$$

Example-Projective Spaces

- Let $x_0 = y_0 = [1 : 0 : \ldots : 0] \in X$. The stabilizers are

$$H_X = \left\{ \begin{pmatrix} a & * \\ 0 & A \end{pmatrix} \in PGL_{d+1} \middle| a \in \mathbb{F}_q^\times \right\}, H_Y = H_X^T.$$

- Let $\psi_X : H_X \to \{\pm 1\}$ and $\psi_Y : H_Y \to \{\pm 1\}$ be defined by

$$\psi_X(B) = \left(\frac{B_{1,1}}{q}\right), \ \psi_Y(C) = \left(\frac{C_{1,1}}{q}\right), \text{ (Legendre Symbols).}$$

- This is enough data to color.
- O_1 is orientable, and O_0 is non-orientable.
- We obtain a $\mathfrak{Coh}^1$ weighing matrix $W\left(\frac{q^{d+1}-1}{q-1}, q^d\right)$.

Orthogonality Without Computations

But why is this matrix (and all Projective/Grassmannian/Flag) orthogonal?

Orthogonality Without Computations

But why is this matrix (and all Projective/Grassmannian/Flag) orthogonal?

Proof:

- If A is a $\mathfrak{Coh}^1$ $X \times Y$ matrix with respect to cocycles (ψ_X, ψ_Y), then AA^* is a $\mathfrak{Coh}^1$ $X \times X$-matrix w.r.t. (ψ_X, ψ_X).
- In our example, $X \times X$ has 2 G-orbits:

$$\Delta = \{(x,x)|x \in X\},$$

$$X \times X \setminus \Delta.$$

- But $X \times X \setminus \Delta$ is non-orientable w.r.t. (ψ_X, ψ_X).
- Hence, AA^* must be 0 off the diagonal! $\square$.

A Hadamard Family

- We shall construct a Hadamard family or order $4q(q+1)$ for every prime-power $q \equiv 1 \mod 4$.
- Consider the Affine Group

$$G = Aff(q) := \{x \mapsto ax + b \mid a \in \mathbb{F}_q^\times, b \in \mathbb{F}_q\}.$$

- Let $Y = \mathbb{F}_q$ with the usual action, and $X = \mathbb{F}_q \times \{0,1\}$ with the following action:

$$g, (x,s) \mapsto \begin{cases} (ax+b, s) & a \in (\mathbb{F}_q^\times)^2 \\ (ax+b, 1-s) & a \notin (\mathbb{F}_q^\times)^2 \end{cases}.$$

- The coefficients will be in $\mu_4 = \{1, -1, i, -i\}$, with G-action by

$$gz = \begin{cases} z & a \in (\mathbb{F}_q^\times)^2 \\ \bar{z} & a \notin (\mathbb{F}_q^\times)^2 \end{cases}.$$

A Hadamard Family

- We use the following two 1-cocycles:

$$\psi_Y = 1$$

$$\psi_X((a,0)) = \begin{cases} 1 & a \in (\mathbb{F}_q^\times)^2 \\ i & a \notin (\mathbb{F}_q^\times)^2 \end{cases}.$$

- The space $X \times Y$ splits into 3 orbits of size $2p, p(p-1), p(p-1)$, all are orientable.
- We create two $2p \times p$ matrices A, B w.r.t. to this coloring, and compute AA^* and BB^*.
- We see that AA^* has four distinct values $\{q, qi, i, -1\}$ at the 1st row. Similarly for BB^* we get $\{q, qi\}$.

A Hadamard Family

- We define matrices $S = [A, \bar{A}]$ and $T = [i\bar{B}, B]$. They satisfy:

$$ST^* + TS^* = 0 \; (ComplexAntiAmicability) \qquad (1)$$

$$qSS^* + TT^* = 2q(q+1)I_{2q}. \qquad (2)$$

- The imaginary part in the Gram cancel alreay at SS^* and TT^*. The -1 in SS^* is repeated q times and cancels the q in TT^*.
- Let C be a classical $(q+1) \times (q+1)$ Symmetric Conference Matrix with diagonal 0.

A Hadamard Family

- We create the matrix

$$U = \begin{pmatrix} T & & C_{i,j}S \\ & \ddots & \\ & & T \end{pmatrix}.$$

- Then $UU^* = 2q(q+1)I_{2q(q+1)}$ and U is a mu_4-Hadamard matrix.
- We replace each entry $z \in \mu_4$ by a 2×2 monomial matrix P_z.
- Finally, we multiply each monomial block by a constant Hadamard $H(2)$, and obtain a classical $H(4q(q+1))$.

The Spectral Sequence

Setting

Let

- G be a finite group,
- X, Y finite G-sets,
- $\mathcal{O} \subset X \times Y$ a G-stable subset,
- μ a G-module.

An Exact Sequence of G-modules

$$\begin{array}{ccccccc} 0 & \longrightarrow & K_{\mathcal{O}} & \longrightarrow & Maps(X,\mu) \oplus Maps(Y,\mu) & \xrightarrow{\delta} & Maps(\mathcal{O},\mu) \\ & & \| & & \| & & \| \\ & & A^0 & & A^1 & & A^2 \end{array}.$$

$\delta(f,g)(x,y)$ ($K_{\mathcal{O}}$) are defined as $f(x)g(y)^{-1}$ ($Ker(\delta)$) resp.

- We understand $Maps(\mathcal{O}, \mu)$ as $X \times Y$-matrices, and $Maps(X, \mu)$ (sim. Y) as diagonal matrices.

Remark

$K_{\mathcal{O}}$ contains μ as G modules by sending $\lambda \mapsto (x \mapsto \lambda, y \mapsto \lambda)$ (constant maps).
Under reasonable conditions (such as when $\mathcal{O} = X \times Y$), $K_{\mathcal{O}} = \mu$.

- Consider the G-module

$$M_D(\mathcal{O}) = coker(\delta) = \mathcal{O} \text{ Matrices modulo } D\text{-equivalence.}$$

Theorem

There is a Spectral Sequence

$$E_1^{p,q} = H^p(G, A^q) \implies H^{p+q-2}(G, M_D(\mathcal{O})).$$

The Filtration

- When $p + q = 2$, the Spectral Sequence Abutes to

$$H^0(G, M_D(\mathcal{O})) = \mathfrak{Coh}(\mathcal{O}, \mu)/D\text{-Equivalence.}$$

- The filtration $F^p H^0(G, M_D(\mathcal{O}))$ agrees with the filtration of $\mathfrak{Coh}(\mathcal{O}, \mu)$ given above. We thus get

Corollary

There are isomorphisms

1. $\mathfrak{Coh}(\mathcal{O}, \mu)/\mathfrak{Coh}^1(\mathcal{O}, \mu) \mod D = E_3^{2,0} \subseteq H^2(G, K_{\mathcal{O}})$.
2. $$\mathfrak{Coh}^1(\mathcal{O}, \mu)/\mathfrak{Coh}^0(\mathcal{O}, \mu) \mod D = E_2^{1,1} = \text{A subquotient of } H^1(G, M_X \oplus M_Y).$$
3. $\mathfrak{Coh}^0(\mathcal{O}, \mu) \mod D = E_3^{0,2} = \text{A quotient of } H^0(G, M_{\mathcal{O}})$.

Comments

- In (2) of the Corollary, we reconstruct the $\mathfrak{Coh}^1$ construction from the 1st cohomology.
- Note a class $c \oplus d \in H^1(G, M_X \oplus M_Y)$ is admissible for a $\mathfrak{Coh}^1$-matrix, if and only if it is in the kernel of

$$H^1(G, M_X \oplus M_Y) \to H^1(G, M_{\mathcal{O}}).$$

- This is exactly the $\mathcal{O}$-Orientability!
- under $c \oplus d$, we still need to choose a class in $H^0(G, M_{\mathcal{O}})$. But this is just Spanning!
- For $\mathfrak{Coh}^2$-matrices, we begin with a class in $H^2(G, K_{\mathcal{O}})$, followed by a choice of a class in $H^1(G, M_X \oplus M_Y)$, followed by a class in $H^0(G, M_{\mathcal{O}})$.

Cocyclic Matrices

Suppose that $X = G$ with the action $g, x \mapsto xg^{-1}$, and $Y = G$ with $g, y \mapsto gy$.

- Then the modules $M_X = M_Y = M_G$ and $M_{\mathcal{O}}$ are G-acyclic, and the Spectral Sequence degenerates to $H^2(G, K_{\mathcal{O}})$ and $H^0(G, M_{\mathcal{O}})$.
- We get only two layers:

$$\mathfrak{Coh}(\mathcal{O}, \mu) \mod D = H^2(G, K_{\mathcal{O}}) \oplus H^0(G, M_{\mathcal{O}})/\mu.$$

- The right summand corresponds to G-developed matrices, and the left summand corresponds to Pure Cocyclic Matrices.
- Note that this also includes cases with nontrivial twist action on μ.

Modified Cocyclic Matrices

Definition

- A Modified G-Developed Matrix is a matrix A of the form

$$A_{g,h} = f(g^{-1}h).$$

- A Modified G-Cocyclic Matrix is a matrix A of the form

$$* \quad A_{g,h} = gf(g^{-1}h)\omega(g^{-1}h, h^{-1})/h\omega(1, h^{-1}).$$

- The reason for this seemingly bizarre definition is that these modified sets are closed under matrix multiplication (when f is Ring Valued), hence they are matrix algebras (see below).

Example $G = Z_4$

- Consider the example from slide 38 with $Z = Z_4$, acting on $X = Y = \{0, 1, 2, 3\}$.
- The $X \times Y$-matrix has 3 orbits: $\begin{pmatrix} 0 & 1 & 2 & 3 \\ 3 & 0 & 1 & 2 \\ 2 & 3 & 0 & 1 \\ 1 & 2 & 3 & 0 \end{pmatrix}$, and is a modified G-develped matrix.
- Define $\mu = \{\pm 1\}, \chi : Z_4 \times Z_4 \mapsto \mu$ by $\chi(x, y) = 1$ for $x + y < 4$ $\chi(x, y) = -1$ for $x + y \geq 4$, where where $0 \leq x, y \leq 3$ and the addition is taken in Z.
- χ satisfies the cocycle formula $\chi(y, z) - \chi(x + y, z) + \chi(x, y + z) - \chi(x, y)$

Example $G = Z_4$ Cont'd

- using for χ formula $*$ above and using the fact that χ is normalized gives the $\mathfrak{Coh}^2$ developed Z_4 matrix

$$X = \begin{pmatrix} o0 & -01 & -02 & -o3 \\ o3 & o0 & o1 & o2 \\ o2 & -o3 & o0 & o1 \\ o1 & -o2 & -o3 & o0 \end{pmatrix}$$

- Setting $o(0,1,2,3)$ equal $0,1,0,1$ respectively means using a $\mathfrak{Coh}^0$ information to the $\mathfrak{Coh}^2, Z_4$ developed matrix .

$$Y = \begin{pmatrix} 0 & -1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & -0 & -1 & 0 \end{pmatrix}$$

- This is a $W(4,2,2)$ matrix.

Example $G = Z_4$ Cont'd

- Define $L = R$ to be the diagonal matrix with terms $(+, -, -, -)$.
- Then $LXR^* = \begin{pmatrix} o0 & 01 & 02 & o3 \\ -o3 & o0 & o1 & o2 \\ -o2 & -o3 & o0 & o1 \\ -o1 & -o2 & -o3 & o0 \end{pmatrix}$
- LXR^* is D equivalent to X and is a $\mathfrak{Coh}^2$ colored Z_4 nega cyclic matrix.
- Setting $o(0, 1, 2, 3)$ to equal $0, 1, 0, 1$ respectively gives a $W(4, 2, 2)$ which is D equivalent, using L and R above, to Y.
- This demonstrates that modified cocyclic matrices are closed under matrix multiplication.

DetFourier Matrix

Here is an example of a $\mathfrak{Coh}^2$-matrix.

Let

- $V = (\mathbb{Z}/N)^2$
- $G = Aff(V) = \{v \mapsto Av + a | A \in GL(2, \mathbb{Z}/N), a \in V\}$.
- $X = Y = V$
- $\mu = \mu_N$ with G-action $\zeta \mapsto \zeta^{\det A}$. Pick a primitive $\zeta_0 \in \mu_N$.
- Consider the 2-cocycle

$$\omega(v \mapsto Av + a, v \mapsto Bv + b) = \zeta_0^{\det(a, Ab)}.$$

- One can compute from the Spectral Sequence a $\mathfrak{Coh}^2$-matrix

$$DF = (\zeta_0^{\det(u,v)}).$$

This is a Hermitian $N^2 \times N^2$ Hadamard Matrix.

A particular DetFourier Matrix

In the particular case $N = 4$ the following matrix is obtained

$$\begin{pmatrix}
1 & 1 & 1 & -1 & i & i & -i & -1 & -i & i & i & -i & -i & 1 & -1 & -1 \\
1 & 1 & 1 & -1 & -i & -i & i & -1 & i & -i & -i & i & i & 1 & -1 & -1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
-1 & -1 & 1 & 1 & -i & i & i & 1 & i & -i & i & -i & -i & 1 & -1 & -1 \\
-i & i & 1 & i & 1 & i & 1 & -i & -1 & -1 & -i & -i & i & -1 & -1 & 1 \\
-i & i & 1 & -i & -i & 1 & i & i & -i & i & -1 & 1 & -1 & -1 & 1 & -1 \\
i & -i & 1 & -i & 1 & -i & 1 & i & -1 & -1 & i & i & -i & -1 & -1 & 1 \\
-1 & -1 & 1 & 1 & i & -i & -i & 1 & -i & i & -i & i & i & 1 & -1 & -1 \\
i & -i & 1 & -i & -1 & i & -1 & i & 1 & 1 & -i & -i & i & -1 & -1 & 1 \\
-i & i & 1 & i & -1 & -i & -1 & -i & 1 & 1 & i & i & -i & -1 & -1 & 1 \\
-i & i & 1 & -i & i & -1 & -i & i & i & -i & 1 & -1 & 1 & -1 & 1 & -1 \\
i & -i & 1 & i & i & 1 & -i & -i & i & -i & -1 & 1 & -1 & -1 & 1 & -1 \\
i & -i & 1 & i & -i & -1 & i & -i & -i & i & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 \\
-1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 \\
-1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1
\end{pmatrix}$$

Graded Hadamard Algebras

Let R be a commutative ring together with a G-action $G \to Aut(R)$, such that $\mu \subset R^\times$.

An Exact Sequence of G-modules

$$\begin{array}{ccccccc} 0 & \longrightarrow & L_{\mathcal{O}} & \longrightarrow & Maps(X, \mu) & \xrightarrow{\delta} & Maps(\mathcal{O}, \mu) \\ & & \| & & \| & & \| \\ & & B^0 & & B^1 & & B^2 \end{array}.$$

where $\delta f(x, y) = f(x)f(y)^{-1}$ and $L_{\mathcal{O}}$ is defined to be the kernel.

Definition

Two square matrices A, B are said to be Δ-Equivalent if

$$A \sim_\Delta B \iff A = DBD^{-1}, D \text{ is diagonal}.$$

Graded Hadamard Algebras

Theorem

There is a spectral sequence

$$H^p(G, B^q) \implies H^{p+q-2}(G, M_\Delta),$$

where M_Δ is $Maps(\mathcal{O}, \mu)/\Delta$ Equivalence.

Definition

A matrix $A \in Maps(\mathcal{O}, \mu)$ is Δ-Cohomology Developed (DCDM) if for all $g \in G$,

$$gA \sim_\Delta A.$$

We denote the group of DCDM's by $\Delta\mathfrak{Coh}(\mathcal{O}, \mu)$.

Graded Hadamard Algebras

- The Spectral Sequence defines a filtration

$$\Delta\mathfrak{Coh}^0(\mathcal{O},\mu) \subseteq \Delta\mathfrak{Coh}^1(\mathcal{O},\mu) \subseteq \Delta\mathfrak{Coh}^2(\mathcal{O},\mu) = \Delta\mathfrak{Coh}(\mathcal{O},\mu),$$

Where

- $\Delta\mathfrak{Coh}^0(\mathcal{O},\mu) = \Delta$-closure of G-invariant matrices,
- $\Delta\mathfrak{Coh}^1(\mathcal{O},\mu)=$ All matrices A such that

$$gA = D(g)AD(g)^{-1}, \text{ where } g \mapsto D(g) \text{ is a 1-cocycle.}$$

- Generally, for $A \in \Delta\mathfrak{Coh}(\mathcal{O},\mu)$, satisfying $gA = D(g)AD(g)^{-1}$, the function $D(g)$ is the coloring data.
- It satisfies: $D(gg') = D(g)g(D(g'))\omega(g,g')$ where ω is a 2-cocycle in $Z^2(G, L_{\mathcal{O}})$.

Coloring Data

Let X be a G-set and $\mathcal{O} \subseteq X \times X$ a stable G-subset.

Definition

A map $D : g \to Maps(G, \mu)$, $g \mapsto D(g)$ is a Coloring Data if is satisfies

$$D(gg') = D(g)g(D(g'))\omega(g, g')$$

where ω is a 2-cocycle in $Z^2(G, L_{\mathcal{O}})$.

By analysis of the above Spectral Sequence one gets:

There is a group $\mathcal{E}$ sitting in the middle of a short exact sequence

$$0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0,$$

where $\mathcal{E}_2 \subseteq H^2(G, L_{\mathcal{O}})$ and $\mathcal{E}_1 \subseteq H^1(G, M_X)/H^1(G, L_{\mathcal{O}})$, and a bijection

$$\mathcal{E} \leftrightarrow \{\text{Coloring Data}\}/L_{\mathcal{O}}.$$

Graded Hadamard Algebras

- Let D be a Coloring Data.
- The set of all matrices $A \in Maps(X \times X, R)$ satisfying

$$gA = D(g)AD(g)^{-1} \quad \forall g \in G,$$

  is closed under matrix multiplication and is an Albegra. Let us denote this algebra by $\mathcal{A}^D$.
- $\mathcal{A}^D$ is closed under conjugate-transpose (*).
- We have

$$\mathcal{A}^D \circ \mathcal{A}^{D'} \subseteq \mathcal{A}^{DD'}.$$

Graded Hadamard Algebra

This motivates

Definition

A Graded Hadamard Algebra (GHA) is a matrix set $\mathcal{H}$ which is a union of matrix algebras

$$\mathcal{H} = \bigcup_{\alpha} \mathcal{H}(\alpha)$$

and satisfies that

$$\mathcal{H}(\alpha) \circ \mathcal{H}(\alpha') \subseteq \mathcal{H}(\beta), \text{ for some } \beta.$$

- The Bose-Mesner Algebra of an association scheme is closed also by Hadamard product, therefore is a GHA.

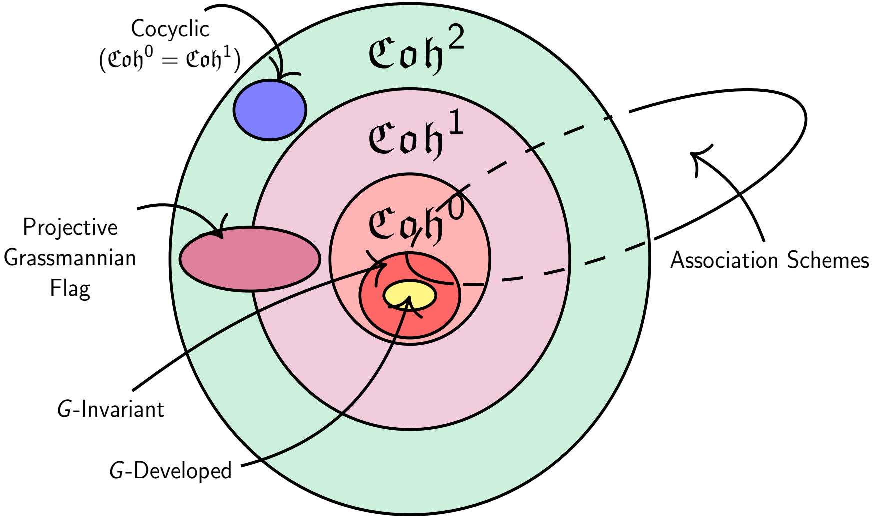
Example - The Brauer Group

- Let L/K be a Galois field extension of degree n with $Gal(L/K) = G$.
- We take $\mu = L^\times$, $X = G$ and $\mathcal{O} = X \times X$. We take $R = L$.
- The group $\mathcal{E} = H^2(G, L^\times)$ (since $H^1(G, M_X) = 0$).
- Every class in $\mathcal{E}$ gives rise to an algebra $\mathcal{A}^D$, which is a central simple algebra over K.
- This gives rise to the well known isomorphism

$$H^2(G, L^\times) = Br(L/K).$$

- We recover the theory of Brauer Groups, where the multiplication is the Hadamard Product.
- Also, elements of the central simple algebras $\mathcal{A}^D$ are Modified Cocyclic Matrices.

A Schemmatic Diagram of the Filtration



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