Constructions of matrices from the developed cohomology filtration

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Vancouver Canadam2019 May 31st

Weighing matrices

- A weighing matrix $W(N,w)$ is a $N \times N$ matrix W whose elements are $0, \pm 1$ such that $WW^T = wI_N$. $W(N,w)$ denotes both a single matrix and the class of all $W(N,w)$.
- The following are $W(2,w), 1 \leq w \leq 2$

$$\left(I_2 \;\middle|\; \begin{matrix} 1 & 1 \\ 1 & - \end{matrix} \right)$$

- The following are $W(3,w), 1 \leq w \leq 3$

$$\left(I_3 \;\middle|\; None \;\middle|\; None \right)$$

- The following are $W(4,w), 1 \leq w \leq 4$

$$\left(\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix} \;\middle|\; \begin{matrix} 1 & 1 & 0 & 0 \\ 1 & - & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & - \end{matrix} \;\middle|\; \begin{matrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & - \\ 1 & - & 0 & 1 \\ - & - & 1 & 0 \end{matrix} \;\middle|\; \begin{matrix} 1 & 1 & 1 & 1 \\ 1 & - & 1 & - \\ 1 & 1 & - & - \\ 1 & - & - & 1 \end{matrix} \right)$$

Weighing matrices

- N is called the order and w is called the weight of W.
- The weight and order satisfiy the inequality $0 \leq w \leq N$.
- $W(N, N)$ denoted $H(N)$ is called a Hadamard matrix.
- Applications: Chemistry, Spectroscopy, Quantum Computing and Coding Theory.
- Main question: Determine the parameters for which a $W(N, w)$ exists.
- The Hadamard Conjecture: $H(N) = W(N, N)$ exists for every $N = 4k, k \in \mathbb{N}$.
- To date the smallest unkonwn Hadamard matrix is $H(668)$.
- The previous unknown Hadamard matrix $H(428)$ was found in 2005 by H. Kharaghani and B. Tayfeh-Rezaiea

Families of weighing matrices

- Sometimes it is possible to find the existence of an infinite family of weighing matrices.
- Some well-known families of Hadamard and weighing matrices. Let q be an odd prime power.
  - Payley Conference Matrices: $W(q+1, q)$
  - Payley Hadamard Matrices: $H(N)$ for $N = q+1$ or $2q+2$.
  - Projective Space Matrices: $W(\frac{q^{d+1}-1}{q-1}, q^d)$.
- In the present work we used cohomology of groups to find one family of (non Hadamard) weighing matrices and 3 families of Hadamard matrices.

Ciculant weighing matrices

- Assuming some extra structure on W may reduce the generality of the construction but also may enable the construction itself.
- One of the classical constructions is of a circulant weighing matrix. [S].
- A matrix is circulant if $W(i,j) = W(i-j,1) \forall i,j, 1 \leq i,j \leq N$ where the subtraction is taken modulu N.
- An example

$$A = \begin{pmatrix} x_0 & x_1 & x_2 & x_3 & x_4 \\ x_4 & x_0 & x_1 & x_2 & x_3 \\ x_3 & x_4 & x_0 & x_1 & x_2 \\ x_2 & x_3 & x_4 & x_0 & x_1 \\ x_1 & x_2 & x_3 & x_4 & x_0 \end{pmatrix}$$

- In this case there are 3^5 possible instead of 3^{25} possible matrices.

An example of a backcirculant weighing matrix

- The following back-circulant matrix becomes a $W(6,4)$ plugging the 1st row $(-,+,0,+,+,0)$ respectively.

$$\begin{pmatrix} o1 & o2 & o3 & o4 & o5 & o6 \\ o2 & o3 & o4 & o5 & o6 & o1 \\ o3 & o4 & o5 & o6 & o1 & o2 \\ o4 & o5 & o6 & o1 & o2 & o3 \\ o5 & o6 & o1 & o2 & o3 & o4 \\ o6 & o1 & o2 & o3 & o4 & o5 \end{pmatrix}, \begin{pmatrix} -1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 & -1 & 1 \\ 1 & 1 & 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 & 1 & 1 \end{pmatrix}$$

An example of a nega-cyclic matrix

- The (back) circulant matrices form too restrictive structure.
- The following is an example of a (back) nega-cyclic structure.

$$W = \begin{pmatrix} o1 & o2 & o3 & o4 & o5 & o6 \\ o2 & o3 & o4 & o5 & o6 & -o1 \\ o3 & o4 & o5 & o6 & -o1 & -o2 \\ o4 & o5 & o6 & -o1 & -o2 & -o3 \\ o5 & o6 & -o1 & -o2 & -o3 & -o4 \\ o6 & -o1 & -o2 & -o3 & -o4 & -o5 \end{pmatrix}$$

- Plugging 1st row $(-,-,-,0,+,-)$ gives $W(6,5)$

$$W = \begin{pmatrix} -1 & -1 & -1 & 0 & 1 & -1 \\ -1 & -1 & 0 & 1 & -1 & 1 \\ -1 & 0 & 1 & -1 & 1 & 1 \\ 0 & 1 & -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & 0 \\ -1 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}$$

Support geometry

- There is a map $W \mapsto |W|$

$$\begin{pmatrix} o1 & o2 & o3 & o4 & o5 & o6 \\ o2 & o3 & o4 & o5 & o6 & -o1 \\ o3 & o4 & o5 & o6 & -o1 & -o2 \\ o4 & o5 & o6 & -o1 & -o2 & -o3 \\ o5 & o6 & -o1 & -o2 & -o3 & -o4 \\ o6 & -o1 & -o2 & -o3 & -o4 & -o5 \end{pmatrix} \mapsto \begin{pmatrix} o1 & o2 & o3 & o4 & o5 & o6 \\ o2 & o3 & o4 & o5 & o6 & o1 \\ o3 & o4 & o5 & o6 & o1 & o2 \\ o4 & o5 & o6 & o1 & o2 & o3 \\ o5 & o6 & o1 & o2 & o3 & o4 \\ o6 & o1 & o2 & o3 & o4 & o5 \end{pmatrix}$$

- Plugging $0, 1$ for all orbits of W gives $|W|$ a $\{0, 1\}$ matrix.
- We view $|W|$ as an incidence matrix of a geometry.
- We abusively call $|W|$ the support geometry of W.
- Our notion of geometry is weaker than the one usually used. There might be several different lines determined by the same set of points.

Cocyclic Matrices

Definition (Group Development)

Let G be a finite group. A matrix A indexed by G is G-developed if it has the form
$$A_{g,h} = f(gh).$$

- For the particular case $G = \mathbb{Z}_n$ this reduces to back circulant matrices of order n.
- It is well known that a G-developed weighing matrix must have weight $w = l^2$, so this is quite restrictive.
- A modification of this is the notion of a Cocyclic Matrix

Definition (Cocyclic Matrices)

A cocyclic G-matrix is a matrix indexed by G of the form
$$B_{g,h} = \omega(g,h)f(gh),$$
where $\omega : G \times G \to \mathbb{C}^{\times}$ is a 2-cocycle.

Start with the geometry

- Cocyclic matrices were developed by Horadam, DeLauney, Flannary,. . . [3]
- For a map $f : G \mapsto \{0,1\}, A$ is the support geometry, in our sense, of B.
- the two cocycle w presents an element in the second cohomology group of $G, H^2(G, \pm 1)$.
- This brings the idea to assume a geometry $|W|$, and use a group G to obtain signs on orbits of the would be matrix W, in order to reduce the enumeration on W.
- We used a similar idea (without assuming a group G) to firstly find a support geometry for (then unknown) $W(23,16)$ and then extend it to all of $W(23,16)$.
- In the 23-16 work we used $S|W| = J - W$ where J is the matrix all of whose terms equal 1,[1]. We call $S|W|$ the shadow geometry of W.

Generalized Weighing Matrices

More generally, let M, N be not necessarily equal natural numbers and

$$\mu_n = \text{ all roots of unity pf order } n, \;\; \mu_n^+ = \mu_n \cup \{0\}.$$

Definition

A Generalized partial Weighing Matrix is a μ_n^+-matrix W such that

$$WW^* = wI_M, \;\; (\text{*=conjugate-transpose}) \;.$$

- Denote $W(M, N, w, n)$ all generalized partial weighing matrices.
- $W(N, N, w, 2)$ gives the classical theory.
- $\mu_1^+ = \{0, 1\}$- are the values for the geometry.

Hadamard Operations and Automprphisms

- Let $Maps(M, N, \mu)$ be the space of all $M \times N$ μ^+-matrices.

Definition

- A Hadamard operation on $Maps(M, N, \mu)$ is given by multiplying both sides with monomial matrices, specifically by
  1. Axis permutations,
  2. Row or column multiplication by signs in μ,
  3. All compositions thereof.
- Denote the group of all Hadamad operations on $Maps(M, N, \mu)$ by $Had(M, N, \mu)$.
- Let $Aut(A)$ be the subgroup of $Had(M, N, \mu)$ preserving A.

- for $n = 1$, $Had(M, N, 1)$ consists only of permutations.
- Given a geometry $|W|$ we choose G to be any subgroup of $Had(M, N, 1)$.

Hadamard equivalence of Weighing Matrices

- A monomial matrix (a signed permutation) P is a permutation matrix whose non zero elements are in μ.
- Two matrices U, V are isomorphic (=Hadamard equivalent) if there exists two monomial matrices P, Q with $PUQ^* = V$.
- The following exhibits an Hadamard equivalence between the Sylvester matrix $H2 \otimes H2$ and the circulant Hadamard matrix $CH4$.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & - \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & - & 1 & - \\ 1 & 1 & - & - \\ 1 & - & - & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ - & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} - & 1 & 1 & 1 \\ 1 & - & 1 & 1 \\ 1 & 1 & - & 1 \\ 1 & 1 & 1 & - \end{bmatrix}$$

Well known classifications of Hadamard equivalence classes

- Another problem is to classify all Hadamard equivalence classes of a possible weighing matrix.
- The existence problem amounts to showing that the number of equivalence classes exceeds 1.
- the following statements concern classical matrices $\mu = \mu_2 = \{\pm 1\}$ and $M = N$.
- There is one class for Hadamrad matrices of each order $N \leq 12$, 5 for $N = 16$, 3 for $N = 20$, and 60 for $N = 24$.
- Chan Rodger Seberry [CRS] classified all $W(N, w)$ for $w \leq 5$ and all $W(N, w)$ for $N \leq 11$.
- They used a method similar to support geometries.
- Harada and Munemasa [HM] classified all $W(N, w)$ for $N \leq 15$, $N = 17$, $W(16, w), w = 6, 9, 12$, and $W(18, 9)$. They found 11891 classes of $W(18, 9)$.

Geometries and Automorphisms

- Hadamard operations on W reduce to permutations on $|W|$. Hence there is a homomorphism

$$Aut(W) \to Aut(|W|)$$

.

- In general this homomorphism is neither injective or surjective.
- We assume a support geometry matrix $|W| \in Maps(M, N, 1)$ and a subgroup $G \subset Aut(|W|)$.
- G has projections $\pi_X(G)$ $(\pi_Y(G))$ - permutation matrices on M (N) elements respectively.
- We use elements $\chi \in H^2(G, \mu), \psi_X \in H^1(\pi_X(G), \mu^M)$ and $\psi_Y \in H^1(\pi_Y(G), \mu^N)$. (We abuse notations, it is $[\chi]$ who is in the cohomology, etc.)

Our Work

- χ, ψ_X and ψ_Y need to satisfy a competablity codition called orientability.
- In the case that $\psi_X = \psi_Y = 0$ orientability is stisfied.
- $X = Y = G$ implies that $\psi_X = \psi_Y = 0$ and our theory reduces to that of the cocyclic matrices:
- The theory includes nega-cyclic matrices as a particular case.
- Finding the action of G on the would be W is called the automorphism lifting problem or also Coloring.
- After coloring the enumeration on W reduces, substituting a single value for each orbit of the action of G on $|W|$.
- Coloring is equivalent to a homomorphism $\overline{\rho} : \hat{G} \mapsto Aut(W)$.
- $W \mapsto |W|$ induces a group homomorphism $\hat{G} \mapsto G$.
- Even if W was found, it need not be a weighing matrix.
- We use a spectral sequence for constructing $\overline{\rho}$
- A lot of trial end error gave us the following new infinite families:

Thank you1

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The Projective space Weighing Matrix Family-Set up

- For every prime-power q integers $0 < d$ and $1 < n|q-1$, define X to be all the lines through the origin in F^{d+1}, Y to be all the d dimensional subspaces of F^{d+1}, and $G = PGL(F, d)$ the quotient of $GL(F, d)$ by the scalar matrices $mI_{d+1}, m \in \mu$.
- G acts faithfully on X and on Y.
- Every choice of a basis in F^{d+1} gives rise to a non degenerate bilinear (tracelike) form $<,> X \times Y \mapsto \mu^{+}$.
- Using this form X and Y are shown to be equivalent G-spaces.
- The $|X| \times |Y|$ incidence matrix defined by $W(l, L) = 1$ if $l \in L$ and $W(l, L) = 0$ otherwise is the support geometry matrix.
- Changing the arbitrary order of elements in X (Y) changes X (Y) respectively by a permutation and does not change the Hadamard equivalence classn of $|W|$.

Projective spaces Weighing Matrices

- Choose χ, ψ_X and ψ_Y carefully.(trial and error).
- In the above setup there is a family of generalized Weighing matrices $W(\frac{q^{d+1}-1}{q-1}, \frac{q^{d+1}-1}{q-1}, q^d, n)$.
- This family equals for $n = 2$ to the family mentioned in the introduction.
- It holds that $\hat{G} = Aut(W) \twoheadrightarrow G = Aut(|W|)$ is surjective.
- For the next family choose also k s.t. $0 < k < d$. The Grasmannian veraiety $X = Gr(F, d, k)$ consists of all the k subspaces in F^{d+1}. Y is set as $G(F, d, d-k)$. Every choice of a basis in F^{d+1} gives rise to a bilinear pairing as above, $G = PGL(F, d)$ acts on the isomorphic G sets X and Y.

The Grassmannian Weighing Matrix Family

- $|X| = |Y| = \begin{bmatrix} d \\ k \end{bmatrix}_q$ the gaussian binomial coeeficient.
- The $|X| \times |Y|$ adjacency matrix equals $dim(I \cap L)$ and for $k \leq \frac{d}{2}$ has $k+1$ possible values.
- The points with a fixed incidence value form an orbit of the G action on the adjacency matrix.
- Setting $W(I, L)$ to equal 1 if $dim(I \cap L)$ above is non zero and 0 otherwise gives a geometry $|W|$ with $k+1$ orbits.
- Choosing smartly χ, ψ_X and ψ_Y gives the following family $W\left(\begin{bmatrix} d \\ k \end{bmatrix}_q, \begin{bmatrix} d \\ k \end{bmatrix}_q, q^{k(d-k)}, n\right)$.
- Again it holds that $\hat{G} = Aut(W) \twoheadrightarrow G = Aut(|W|)$ is surjective.
- Substututing $k = 1, d = d + 1$ reduces the Grassmannian family to the above classical projective space family.

Set up for the Flag varaiety Weighing Matrix Family

- Given $r \in \mathbb{N}$ and $0 < k_0 \leq k_1 \leq \cdots \leq k_r$ integers, s.t $d = \sum k_i$, an r flag is a sequence of subspaces $K_0 \subseteq K_1 \subseteq K_2 \cdots \subseteq K_r = F^{d+1}$, s.t $dim(K_i) = \sum_{j=0,i} k_j$.
- In the case $r = 1$ we denote (after possibly reordering k and $d - k$) $k_0 = k$, $k_1 = d - k$ and recover the setup for the Grassmannians.
- Observe that $r = 1$ satisfies the inequality $r \leq n$.

Theorem (Flag Variety Weighing Matrices)

Let q be a prime-power, let $0 < k_0 \leq k_1 \leq \cdots \leq k_r$ be integers, let $d = \sum k_i$, and let $n | q - 1$. Assume that $n \geq r$. Then there exists a generalized weighing matrix,

$$GW\left(\begin{bmatrix} d \\ k_0, k_1, \ldots, k_r \end{bmatrix}_q, \begin{bmatrix} d \\ k_0, k_1, \ldots, k_r \end{bmatrix}_q, q^{\sum_{i<j} k_i k_j}, n\right),$$

Again $Aut(W) \twoheadrightarrow Aut(|W|)$.

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Quasi Projective/Grassmannian/Flag Weighing Matrices

Theorem (Quasi Projective/Grassmannian/Flag Weighing Matrices)

Assume that the conditions of the previous theorem are satisfied and $GW(N, w, n)$ is the matrix guaranteed by the theorem. Suppose that a circulant weighing matrix $CW(M, t, n)$ exists, for some $M | q-1$. Then there can be constructed a generalized

$$W = GW(MN, tw, n), \quad Aut(W) \twoheadrightarrow Aut(|W|),$$

and in general $W \not\approx GW(N, n, n) \otimes CW(M, t, n)$.

Remarks:

1. This construction is sort of a Twisted Kronnecker Product.
2. W cannot be obtained from both matrices by Craigen's weaving technique, as can be shown by computing a certain invariant.

Three families of Hadamard matrices

Theorem

for every prime power $q \equiv 1 \mod 4$ *, there exists an Hadamard marix* $H(4q(q+1))$.

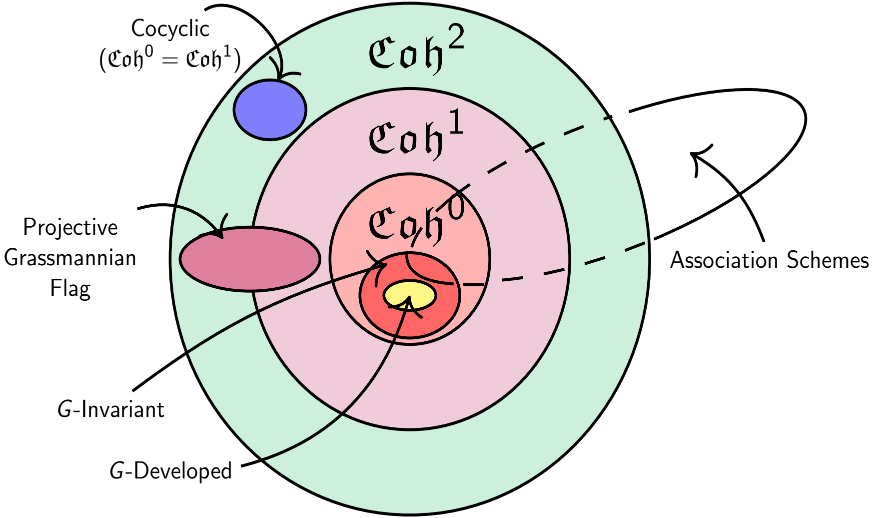
Theorem

for every prime power $q \equiv 3 \mod 4$, *such that* $q-4$ *is a prime power too , there exists an Hadamard marix* $H(8q(q-3))$.

Theorem

for every prime power $q \equiv 3 \mod 4$, *such that* $q-2$ *is a prime power too , there exists an Hadamard marix* $H(8q(q-1))$.

A Schemmatic Diagram of the Filtration



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- Algebraic Design Theory (Mathematical Surveys and Monographs) by Warwick de Launey (Author), Dane Flannery (Author) Publication Year: 2011 ISBN-10: 0-8218-4496-2 ISBN-13: 978-0-8218-4496-0 https://www.ams.org/publications/authors/books/postpub/surv-175

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